(*b*) Show that the mapping $T : \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $T(\boldsymbol{A}) = \boldsymbol{A} - \boldsymbol{A}^{\boldsymbol{T}}$ is a linear operator on $\mathcal{M}_{nn}$.

5. Let $\boldsymbol{P}$ be a fixed non-singular matrix in $\mathcal{M}_{nn}$. Show that the mapping $T : \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $T(\boldsymbol{A}) = \boldsymbol{P}^{-1}\boldsymbol{AP}$ is a linear operator.

6. Let V and W be vector spaces. Show that a function $T : V \to W$ is a linear transformation if and only if $T(\alpha \boldsymbol{v_1} + \beta \boldsymbol{v_2}) = \alpha T(\boldsymbol{v_1}) + \beta T(\boldsymbol{v_2})$, for all $\boldsymbol{v_1}, \boldsymbol{v_2} \in V$ and all $\alpha, \beta \in \mathbb{R}$.

7. Let $T_1, T_2 : V \to W$ be linear transformations. Define

$$T_1 + T_2 : V \to W \text{ by } \quad (T_1 + T_2)(\boldsymbol{v}) = T_1(\boldsymbol{v}) + T_2(\boldsymbol{v}), \ \boldsymbol{v} \in V$$

Also, define

$$cT_1 : V \to W \text{ by } \quad (cT_1)(\boldsymbol{v}) = c(T_1(\boldsymbol{v})), \ \boldsymbol{v} \in V$$

Show that $T_1 + T_2$ and cT are linear transformations.

ANSWERS

1. (*a*) *Yes* (*b*) *No* (*c*) *No* (*d*) *No* (*e*) *Yes* (*f*) *No* (*g*) *No*

6.2 THE MATRIX OF A LINEAR TRANSFORMATION

In this section we will show that a linear transformation between finite-dimensional vector spaces is uniquely determined if we know its action on an ordered basis for the domain. We will also show that every linear transformation between finite-dimensional vector spaces has a unique matrix $\boldsymbol{A}_{BC}$ with respect to the ordered bases B and C chosen for the domain and codomain, respectively.

A Linear Transformation is Determined by its Action on a Basis

One of the most useful properties of linear transformations is that, if we know how a linear map $T : V \to W$ acts on a basis of V, then we know how it acts on the whole of V.

THEOREM 6.4 Let $B = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n}\}$ be an ordered basis for a vector space V. Let W be a vector space, and let $\boldsymbol{w_1}, \boldsymbol{w_2}, ..., \boldsymbol{w_n}$ be any n (not necessarily distinct) vectors in W. Then there is one and only one linear transformation $T : V \to W$ satisfying $T(\boldsymbol{v_1}) = \boldsymbol{w_1}, T(\boldsymbol{v_2}) = \boldsymbol{w_2}, ..., T(\boldsymbol{v_n}) = \boldsymbol{w_n}$.
In other words, a linear transformation is determined by its action on a basis.

Proof Let $\boldsymbol{v}$ be any vector in V. Since $B = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n}\}$ is an ordered basis for V, there exist unique scalars $a_1, a_2, ..., a_n$ in $\mathbb{R}$ such that $\boldsymbol{v} = a_1 \boldsymbol{v_1} + a_2 \boldsymbol{v_2} + ... + a_n \boldsymbol{v_n}$.
Define a function $T : V \to W$ by

$$T(\boldsymbol{v}) = a_1 \boldsymbol{w_1} + a_2 \boldsymbol{w_2} + ... + a_n \boldsymbol{w_n}$$

Since the scalars a_i's are unique, T is well-defined. We will show that T is a linear transformation. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two vectors in V. Then

$$\boldsymbol{x} = b_1 \boldsymbol{v_1} + b_2 \boldsymbol{v_2} + ... + b_n \boldsymbol{v_n}$$

and $\quad \boldsymbol{y} = c_1 \boldsymbol{v_1} + c_2 \boldsymbol{v_2} + ... + c_n \boldsymbol{v_n}$

for some unique b_i's and c_i's in $\mathbb{R}$. Then, by definition of T, we have

$$T(\boldsymbol{x}) = b_1 \boldsymbol{w_1} + b_2 \boldsymbol{w_2} + ... + b_n \boldsymbol{w_n}$$

$$T(\boldsymbol{y}) = c_1 \boldsymbol{w_1} + c_2 \boldsymbol{w_2} + ... + c_n \boldsymbol{w_n}$$

$$\therefore \quad T(\boldsymbol{x}) + T(\boldsymbol{y}) = (b_1 \boldsymbol{w_1} + b_2 \boldsymbol{w_2} + ... + b_n \boldsymbol{w_n}) + (c_1 \boldsymbol{w_1} + c_2 \boldsymbol{w_2} + ... + c_n \boldsymbol{w_n})$$

$$= (b_1 + c_1)\boldsymbol{w_1} + (b_2 + c_2)\boldsymbol{w_2} + ... + (b_n + c_n)\boldsymbol{w_n}$$

However, $$\boldsymbol{x} + \boldsymbol{y} = (b_1 \boldsymbol{v_1} + b_2 \boldsymbol{v_2} + ... + b_n \boldsymbol{v_n}) + (c_1 \boldsymbol{v_1} + c_2 \boldsymbol{v_2} + ... + c_n \boldsymbol{v_n})$$

$$= (b_1 + c_1)\boldsymbol{v_1} + (b_2 + c_2)\boldsymbol{v_2} + ... + (b_n + c_n)\boldsymbol{v_n}$$

$$\therefore \quad T(\boldsymbol{x} + \boldsymbol{y}) = (b_1 + c_1)\boldsymbol{w_1} + (b_2 + c_2)\boldsymbol{w_2} + ... + (b_n + c_n)\boldsymbol{w_n},$$

again by definition of T. Hence, $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$. Next, for any scalar $c \in \mathbb{R}$,

$$c\boldsymbol{x} = c(b_1 \boldsymbol{v_1} + b_2 \boldsymbol{v_2} + ... + b_n \boldsymbol{v_n}) = (cb_1)\boldsymbol{v_1} + (cb_2)\boldsymbol{v_2} + ... + (cb_n)\boldsymbol{v_n}$$

$$\Rightarrow \quad T(c\boldsymbol{x}) = (cb_1)\boldsymbol{w_1} + (cb_2)\boldsymbol{w_2} + ... + (cb_n)\boldsymbol{w_n}$$

$$= c(b_1 \boldsymbol{w_1}) + c(b_2 \boldsymbol{w_2}) + ... + c(b_n \boldsymbol{w_n})$$

$$= c(b_1 \boldsymbol{w_1} + b_2 \boldsymbol{w_2} + ... + b_n \boldsymbol{w_n})$$

$$= cT(\boldsymbol{x})$$

Hence T is a linear transformation.

To prove the uniqueness, let $L : V \rightarrow W$ be another linear transformation satisfying

$$L(\boldsymbol{v_1}) = \boldsymbol{w_1}, \; L(\boldsymbol{v_2}) = \boldsymbol{w_2}, \; ..., L(\boldsymbol{v_n}) = \boldsymbol{w_n}$$

If $\boldsymbol{v} \in V$, then $\boldsymbol{v} = a_1 \boldsymbol{v_1} + a_2 \boldsymbol{v_2} + ... + a_n \boldsymbol{v_n}$, for unique scalars $a_1, a_2, ..., a_n \in \mathbb{R}$. But then

$$L(\boldsymbol{v}) = L(a_1 \boldsymbol{v_1} + a_2 \boldsymbol{v_2} + ... + a_n \boldsymbol{v_n})$$

$$= a_1 L(\boldsymbol{v_1}) + a_2 L(\boldsymbol{v_2}) + ... + a_n L(\boldsymbol{v_n}) \quad (\because \; L \text{ is a L.T.})$$

$$= a_1 \boldsymbol{w_1} + a_2 \boldsymbol{w_2} + ... + a_n \boldsymbol{w_n} = T(\boldsymbol{v})$$

$\Rightarrow \quad L = T$ and hence T is uniquely determined.

EXAMPLE 14 Suppose $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear transformation with

$$L([1, -1, 0]) = [2, 1], L([0, 1, -1]) = [-1, 3] \text{ and } L([0, 1, 0]) = [0, 1].$$

Find $L([-1, 1, 2])$. Also, give a formula for $L([x, y, z])$, for any $[x, y, z] \in \mathbb{R}^3$.

[*Delhi Univ. GE*-2, 2017]

SOLUTION To find $L([-1, 1, 2])$, we need to express the vector $\boldsymbol{v} = [-1, 1, 2]$ as a linear combination of vectors $\boldsymbol{v_1} = [1, -1, 0]$, $\boldsymbol{v_2} = [0, 1, -1]$ and $\boldsymbol{v_3} = [0, 1, 0]$. That is, we need to find constants a_1, a_2 and a_3 such that

$$\boldsymbol{v} = a_1 \boldsymbol{v_1} + a_2 \boldsymbol{v_2} + a_3 \boldsymbol{v_3},$$

which leads to the linear system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 2 \end{array}\right]$$

We transform this matrix to reduced row echelon form :

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

This gives $a_1 = -1$, $a_2 = -2$, and $a_3 = 2$. So,

$$\boldsymbol{v} = -\boldsymbol{v_1} - 2\boldsymbol{v_2} + 2\boldsymbol{v_3}$$

$$\Rightarrow \quad L(\boldsymbol{v}) = L(-\boldsymbol{v_1} - 2\boldsymbol{v_2} + 2\boldsymbol{v_3})$$

$$= L(-\boldsymbol{v_1}) - 2L(\boldsymbol{v_2}) + 2L(\boldsymbol{v_3})$$

$$= -[2, 1] - 2[-1, 3] + 2[0, 1] = [0, -5]$$

i.e., $L([-1, 1, 2]) = [0, -5]$

To find $L([x, y, z])$ for any $[x, y, z] \in \mathbb{R}^3$, we row reduce

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ -1 & 1 & 1 & y \\ 0 & -1 & 0 & z \end{array}\right] \quad \text{to obtain} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & -z \\ 0 & 0 & 1 & x+y+z \end{array}\right]$$

Thus, $[x, y, z] = x\,\boldsymbol{v_1} - z\,\boldsymbol{v_2} + (x + y + z)\,\boldsymbol{v_3}$

$$\Rightarrow \quad L([x, y, z]) = L(x\,\boldsymbol{v_1} - z\,\boldsymbol{v_2} + (x + y + z)\,\boldsymbol{v_3})$$

$$= x\,L(\boldsymbol{v_1}) - z\,L(\boldsymbol{v_2}) + (x + y + z)L(\boldsymbol{v_3})$$

$$= x[2, 1] - z[-1, 3] + (x + y + z)\,[0, 1]$$

$$= [2x + z, 2x + y - 2z].$$

EXAMPLE 15 Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator and $L([1, 1]) = [1, -3]$ and $L([-2, 3]) = [-4, 2]$. Express $L([1, 0])$ and $L([0, 1])$ as linear combinations of the vectors [1, 0] and [0, 1]. **[*Delhi Univ. GE*-2, 2019]**

SOLUTION To find $L([1, 0])$ and $L([0, 1])$, we first express the vectors $\boldsymbol{v_1} = [1, 0]$ and $\boldsymbol{v_2} = [0, 1]$ as linear combinations of vectors $\boldsymbol{w_1} = [1, 1]$ and $\boldsymbol{w_2} = [-2, 3]$. To do this, we row reduce the augmented matrix

$$[\boldsymbol{w_1}\ \boldsymbol{w_2} \mid \boldsymbol{v_1}\ \boldsymbol{v_2}] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

Thus, we row reduce

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 5 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{5}R_2} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -1/5 & 1/5 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 3/5 & 2/5 \\ 0 & 1 & -1/5 & 1/5 \end{array}\right]$$

$$\Rightarrow \qquad \boldsymbol{v_1} = \frac{3}{5}\boldsymbol{w_1} - \frac{1}{5}\boldsymbol{w_2} \quad \text{and} \quad \boldsymbol{v_2} = \frac{2}{5}\boldsymbol{w_1} + \frac{1}{5}\boldsymbol{w_2}$$

This gives
$$\begin{aligned} L(\boldsymbol{v_1}) &= \frac{3}{5}L(\boldsymbol{w_1}) - \frac{1}{5}L(\boldsymbol{w_2}) \\ &= \frac{3}{5}L([1,\ 1]) - \frac{1}{5}L([-2,\ 3]) \\ &= \frac{3}{5}[1, -3] - \frac{1}{5}[-4, 2] = \left[\frac{7}{5},\ \frac{-11}{5}\right] = \frac{7}{5}[1,\ 0] - \frac{11}{5}[0,\ 1] \end{aligned}$$

and
$$\begin{aligned} L(\boldsymbol{v_2}) &= \frac{2}{5}L(\boldsymbol{w_1}) + \frac{1}{5}L(\boldsymbol{w_2}) \\ &= \frac{2}{5}L([1,\ 1]) + \frac{1}{5}L([-2,\ 3]) \\ &= \frac{2}{5}[1, -3] + \frac{1}{5}[-4, 2] = \left[\frac{-2}{5},\ \frac{-4}{5}\right] = -\frac{2}{5}[1,\ 0] - \frac{4}{5}[0,\ 1]. \end{aligned}$$

The Matrix of a Linear Transformation

We now show that any linear transformation on a finite-dimensional vector space can be expressed as a matrix multiplication. This will enable us to find the effect of any linear transformation by simply using matrix multiplication.

Let V and W be non-trivial vector spaces, with dim $V = n$ and dim $W = m$. Let $B = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n}\}$ and $C = \{\boldsymbol{w_1}, \boldsymbol{w_2}, ..., \boldsymbol{w_m}\}$ be ordered bases for V and W, respectively. Let $T : V \to W$ be a linear transformation. For each $\boldsymbol{v}$ in V, the coordinate vectors for $\boldsymbol{v}$ and $T(\boldsymbol{v})$ with respect to ordered bases B and C are $[\boldsymbol{v}]_B$ and $[T(\boldsymbol{v})]_C$, respectively. Our goal is to find an $m \times n$ matrix $\boldsymbol{A} = (a_{ij})$ $(1 \le i \le m\ ;\ 1 \le j \le n)$ such that

$$\boldsymbol{A}[\boldsymbol{v}]_B = [T(\boldsymbol{v})]_C \qquad ...(1)$$

holds for all vectors $\boldsymbol{v}$ in V. Since Equation (1) must hold for all vectors in V, it must hold, in particular, for the basis vectors in B, that is,

$$\boldsymbol{A}[\boldsymbol{v_1}]_B = [T(\boldsymbol{v_1})]_C,\ \boldsymbol{A}[\boldsymbol{v_2}]_B = [T(\boldsymbol{v_2})]_C,\ ....,\ \boldsymbol{A}[\boldsymbol{v_n}]_B = [T(\boldsymbol{v_n})]_C \qquad ...(2)$$

But $\quad [\boldsymbol{v_1}]_B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad [\boldsymbol{v_2}]_B = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad ...., \quad [\boldsymbol{v_n}]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$

$$\therefore \qquad \boldsymbol{A}[\boldsymbol{v_1}]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

$$\boldsymbol{A}[\boldsymbol{v_2}]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$$

$$\boldsymbol{A}[\boldsymbol{v_n}]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Substituting these results into (2), we obtain

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = [T(\boldsymbol{v_1})]_C, \quad \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} = [T(\boldsymbol{v_2})]_C, \quad ......, \quad \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = [T(\boldsymbol{v_n})]_C,$$

This shows that the successive columns of $\boldsymbol{A}$ are the coordinate vectors of $T(\boldsymbol{v_1})$, $T(\boldsymbol{v_2})$, ..., $T(\boldsymbol{v_n})$ with respect to the ordered basis C. Thus, the matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = [[T(\boldsymbol{v_1})]_C \quad [T(\boldsymbol{v_2})]_C \qquad [T(\boldsymbol{v_n})]_C]$$

We will call this matrix as the **matrix of *T* relative to the bases *B* and *C*** and will denote it by the symbol $\boldsymbol{A_{BC}}$ or $[\boldsymbol{T}]_{\boldsymbol{BC}}$. Thus,

$$\boldsymbol{A_{BC}} = [[T(\boldsymbol{v_1})]_C \quad [T(\boldsymbol{v_2})]_C \ \ldots \ T(\boldsymbol{v_n})]_C]$$

From (1), the matrix $\boldsymbol{A_{BC}}$ satisfies the property

$$\boldsymbol{A_{BC}}\,[\boldsymbol{v}]_{\boldsymbol{B}} = [T(\boldsymbol{v})]_{\boldsymbol{C}} \text{ for all } \boldsymbol{v} \in V.$$

We have thus proved :

THEOREM 6.5 Let V and W be non-trivial vector spaces, with $\dim(V) = n$ and $\dim(W) = m$. Let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ and $C = \{\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_m\}$ be ordered bases for V and W, respectively. Let $T : V \to W$ be a linear transformation. Then there is a unique $m \times n$ matrix $\boldsymbol{A_{BC}}$ such that $\boldsymbol{A_{BC}}[\boldsymbol{v}]_{\boldsymbol{B}} = [T[\boldsymbol{v}]]_C$, for all $\boldsymbol{v} \in V$. (That is $\boldsymbol{A_{BC}}$ times the coordinatization of $\boldsymbol{v}$ with respect to $\boldsymbol{B}$ gives the coordinatization of $T(\boldsymbol{v})$ with respect to C).

Furthermore, for $1 \le \boldsymbol{i} \le \boldsymbol{n}$, the $\boldsymbol{i}$th column of $\boldsymbol{A_{BC}} = [T[\boldsymbol{v_i}]]_C$.

EXAMPLE 16 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by $T([x_1, x_2, x_3]) = [3x_1 + x_2, x_1 + x_3, x_1 - x_3]$. Find the matrix for T with respect to the standard basis for $\mathbb{R}^3$.

SOLUTION The standard basis for $\mathbb{R}^3$ is $B = \{\boldsymbol{e_1} = [1, 0, 0], \boldsymbol{e_2} = [0, 1, 0], \boldsymbol{e_3} = [0, 0, 1]\}$. Substituting each standard basis vector into the given formula for T shows that

$$T(\boldsymbol{e_1}) = [3, 1, 1], \quad T(\boldsymbol{e_2}) = [1, 0, 0], \quad T(\boldsymbol{e_3}) = [0, 1, -1]$$

Since the coordinate vector of any element $[x_1, x_2, x_3]$ in $\mathbb{R}^3$ with respect to the standard basis $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we have

$$[T(\boldsymbol{e_1})]_B = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \qquad [T(\boldsymbol{e_2})]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [T(\boldsymbol{e_3})]_B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Thus, the matrix $\boldsymbol{A_{BB}}$ for T with respect to the standard basis is :

$$\boldsymbol{A_{BB}} = [[T(\boldsymbol{e_1})]_B \;\; [T(\boldsymbol{e_2})]_B \;\; [T(\boldsymbol{e_3})]_B] = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

EXAMPLE 17 Let $T : \mathcal{P}_3 \to \mathbb{R}^3$ be the linear transformation given by $T(ax^3 + bx^2 + cx + d) = [4a - b + 3c + 3d, a + 3b - c + 5d, -2a - 7b + 5c - d]$. Find the matrix for T with respect to the standard bases $B = \{x^3, x^2, x, 1\}$ for $\mathcal{P}_3$ and $C = \{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ for $\mathbb{R}^3$.

SOLUTION Substituting each standard basis vector in B into the given formula for T shows that $T(x^3) = [4, 1, -2]$, $T(x^2) = [-1, 3, -7]$, $T(x) = [3, -1, 5]$ and $T(1) = [3, 5, -1]$. Since we are using the standard basis C for $\mathbb{R}^3$,

$$[T(x^3)]_C = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}, \qquad [T(x^2)]_C = \begin{bmatrix} -1 \\ 3 \\ -7 \end{bmatrix}, \qquad [T(x)]_C = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}, \qquad [T(1)]_C = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$$

Thus, the matrix of T with respect to the bases B and C is:

$$\boldsymbol{A}_{BC} = [[T(x^3)]_C \;\; [T(x^2)]_C \;\; [T(x)]_C \;\; [T(1)]_C] = \begin{bmatrix} 4 & -1 & 3 & 3 \\ 1 & 3 & -1 & 5 \\ -2 & -7 & 5 & -1 \end{bmatrix}.$$

EXAMPLE 18 Let $T : \mathcal{P}_3 \to \mathcal{P}_2$ be the linear transformation given by $T(\boldsymbol{p}) = \boldsymbol{p}'$, where $\boldsymbol{p} \in \mathcal{P}_3$. Find the matrix for T with respect to the standard bases for $\mathcal{P}_3$ and $\mathcal{P}_2$. Use this matrix to calculate $T(4x^3 - 5x^2 + 6x - 7)$ by matrix multiplication.

SOLUTION The standard basis for $\mathcal{P}_3$ is $B = \{x^3, x^2, x, 1\}$, and the standard basis for $\mathcal{P}_2$ is $C = \{x^2, x, 1\}$. Computing the derivative of each polynomial in the standard bases B for $\mathcal{P}_3$ shows that

$$T(x^3) = 3x^2, \quad T(x^2) = 2x, \quad T(x) = 1, \text{ and } T(1) = 0.$$

We convert these resulting polynomials in $\mathcal{P}_2$ to vectors in $\mathbb{R}^3$:

$$3x^2 \to [3, 0, 0]; \quad 2x \to [0, 2, 0]; \quad 1 \to [0, 0, 1]; \quad \text{and} \quad 0 \to [0, 0, 0].$$

Using each of these vectors as columns yields

$$\boldsymbol{A}_{BC} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We will compute $T(4x^3 - 5x^2 + 6x - 7)$ using this matrix. Now,

$$[4x^3 - 5x^2 + 6x - 7]_B = \begin{bmatrix} 4 \\ -5 \\ 6 \\ -7 \end{bmatrix}$$

Hence,

$$[T(4x^3 - 5x^2 + 6x - 7)]_C = \boldsymbol{A}_{BC}[4x^3 - 5x^2 + 6x - 7]_B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ -5 \\ 6 \\ -7 \end{bmatrix} = \begin{bmatrix} 12 \\ -10 \\ 6 \end{bmatrix}.$$

Converting back from C-coordinates to polynomials gives

$$T(4x^3 - 5x^2 + 6x - 7) = 12x^2 - 10x + 6.$$

EXAMPLE 19 Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $T([x_1, x_2, x_3]) = [-2x_1 + 3x_3, x_1 + 2x_2 - x_3]$. Find the matrix for T with respect to the ordered bases $B = \{[1, -3, 2], [-4, 13, -3], [2, -3, 20]\}$ for $\mathbb{R}^3$ and $C = \{[-2, -1], [5, 3]\}$ for $\mathbb{R}^2$. **[*Delhi Univ. GE*-2, 2019(*Modified*)]**

SOLUTION By definition, the matrix $\boldsymbol{A}_{BC}$ of T with respect to the ordered bases B and C is given by $\boldsymbol{A}_{BC} = [[\mathrm{T}(\boldsymbol{v}_1)]_C \ [\mathrm{T}(\boldsymbol{v}_2)]_C \ [\mathrm{T}(\boldsymbol{v}_3)]_C]$, where $\boldsymbol{v}_1 = [1, -3, 2]$, $\boldsymbol{v}_2 = [-4, 13, -3]$, and $\boldsymbol{v}_3 = [2, -3, 20]$ are the basis vectors in B. Substituting each basis vector in B into the given formula for T shows that

$$T(\boldsymbol{v}_1) = [4, -7], \quad T(\boldsymbol{v}_2) = [-1, 25], \quad T(\boldsymbol{v}_3) = [56, -24]$$

Next, we must find the coordinate vector of each of these images in $\mathbb{R}^2$ with respect to the C basis. To do this, we use the Coordinatization Method. Thus, we must row reduce matrix

$$[\boldsymbol{w}_1 \quad \boldsymbol{w}_2 \mid T(\boldsymbol{v}_1) \quad T(\boldsymbol{v}_2) \quad T(\boldsymbol{v}_3)],$$

where $\boldsymbol{w}_1 = [-2, -1]$, $\boldsymbol{w}_2 = [5, 3]$ are the basis vectors in C. Thus, we row reduce

$$\left[\begin{array}{cc|ccc} -2 & 5 & 4 & -1 & 56 \\ -1 & 3 & -7 & 25 & -24 \end{array}\right] \text{ to obtain } \left[\begin{array}{cc|ccc} 1 & 0 & -47 & 128 & -288 \\ 0 & 1 & -18 & 51 & -104 \end{array}\right]$$

Hence $[T(\boldsymbol{v}_1)]_C = \begin{bmatrix} -47 \\ -18 \end{bmatrix}, \quad [T(\boldsymbol{v}_2)]_C = \begin{bmatrix} 128 \\ 51 \end{bmatrix}, \quad [T(\boldsymbol{v}_3)]_C = \begin{bmatrix} -288 \\ -104 \end{bmatrix}$

$\therefore$ The matrix of T with respect to the bases B and C is $\boldsymbol{A}_{BC} = \begin{bmatrix} -47 & 128 & -288 \\ -18 & 51 & -104 \end{bmatrix}$.

Finding the New Matrix for a Linear Transformation After a Change of Basis

We now state a theorem (proof omitted) which helps us in computing the matrix for a linear transformation when we change the bases for the domain and codomain.

THEOREM 6.6 Let V and W be two non-trivial finite-dimensional vector spaces with ordered bases B and C, respectively. Let $T : V \to W$ be a linear transformation with matrix $\boldsymbol{A}_{BC}$ with respect to bases B and C. Suppose that D and E are other ordered bases for V and W, respectively. Let $\boldsymbol{P}$ be the transition matrix from B to D, and let $\boldsymbol{Q}$ be the transition matrix from C to E. Then the matrix $\boldsymbol{A}_{DE}$ for T with respect to bases D and E is given by $\boldsymbol{A}_{DE} = \boldsymbol{Q}\boldsymbol{A}_{BC}\boldsymbol{P}^{-1}$.

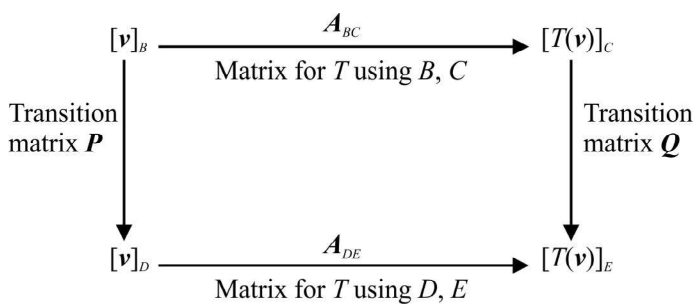

FIGURE 6.6 Illustrates the situation in Theorem 6.6

EXAMPLE 20 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by $T[(a, b, c)] = [-2a + b, -b - c, a + 3c]$.

(a) Find the matrix $\boldsymbol{A}_{BB}$ for T with respect to the standard basis $B = \{\boldsymbol{e}_1 = [1, 0, 0], \boldsymbol{e}_2 = [0, 1, 0], \boldsymbol{e}_3 = [0, 0, 1]\}$ for $\mathbb{R}^3$.

(b) Use part (a) to find the matrix $\boldsymbol{A}_{DE}$ with respect to the standard bases $D = \{[15, -6, 4], [2, 0, 1], [3, -1, 1]\}$ and $E = \{[1, -3, 1], [0, 3, -1], [2, -2, 1]\}$.

SOLUTION (a) We have $T(\boldsymbol{e}_1) = [-2, 0, 1]$, $T(\boldsymbol{e}_2) = [1, -1, 0]$, $T(\boldsymbol{e}_3) = [0, -1, 3]\}$. Using each of these vectors as columns yields the matrix $\boldsymbol{A}_{BB}$:

$$\boldsymbol{A}_{BB} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix}$$

(*b*) To find $\boldsymbol{A}_{DE}$, we make use of the following relationship :

$$\boldsymbol{A}_{DE} = \boldsymbol{Q}\boldsymbol{A}_{BB}\,\boldsymbol{P}^{-1}, \qquad \text{...(1)}$$

where $\boldsymbol{P}$ is the transition matrix from B to D and $\boldsymbol{Q}$ is the transition matrix from B to E. Since $\boldsymbol{P}^{-1}$ is the transition matrix from D to B and B is the standard basis for $\mathbb{R}^3$, it is given by

$$\boldsymbol{P}^{-1} = \begin{bmatrix} 15 & 2 & 3 \\ -6 & 0 & -1 \\ 4 & 1 & 1 \end{bmatrix}$$

To find $\boldsymbol{Q}$, we first find $\boldsymbol{Q}^{-1}$, the transition matrix from E to B, which is given by

$$\boldsymbol{Q}^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

It can be easily checked that

$$\boldsymbol{Q} = (\boldsymbol{Q}^{-1})^{-1} = \begin{bmatrix} 1 & -2 & -6 \\ 1 & -1 & -4 \\ 0 & 1 & 3 \end{bmatrix}$$

Hence, using Eq. (1), we obtain

$$\boldsymbol{A}_{DE} = \boldsymbol{Q}\boldsymbol{A}_{BB}\,\boldsymbol{P}^{-1} = \begin{bmatrix} 1 & -2 & -6 \\ 1 & -1 & -4 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 15 & 2 & 3 \\ -6 & 0 & -1 \\ 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -202 & -32 & -43 \\ -146 & -23 & -31 \\ 83 & 14 & 18 \end{bmatrix}.$$

EXAMPLE 21 Let $T : \mathcal{P}_3 \to \mathbb{R}^3$ be the linear transformation given by $T(ax^3 + bx^2 + cx + d) = [c + d, 2b, a - d]$.

(*a*) Find the matrix $\boldsymbol{A}_{BC}$ for T with respect to the standard bases B (for $\mathcal{P}_3$) and C (for $\mathbb{R}^3$).

(*b*) Use part (*a*) to find the matrix $\boldsymbol{A}_{DE}$ for T with respect to the standard bases $D = \{x^3 + x^2, x^2 + x, x + 1, 1\}$ for $\mathcal{P}_3$ and $E = \{[-2, 1, -3], [1, -3, 0], [3, -6, 2]\}$ for $\mathbb{R}^3$.

SOLUTION (*a*) To find the matrix $\boldsymbol{A}_{BC}$ for T with respect to the standard bases $B = \{x^3, x^2, x, 1\}$ for $\mathcal{P}_3$ and $C = \{\boldsymbol{e}_1 = [1, 0, 0], \boldsymbol{e}_2 = [0, 1, 0], \boldsymbol{e}_3 = [0, 0, 1]\}$ for $\mathbb{R}^3$, we first need to find $T(\boldsymbol{v})$ for each $\boldsymbol{v} \in B$. By definition of T, we have

$$T(x^3) = [0, 0, 1], \quad T(x^2) = [0, 2, 0], \quad T(x) = [1, 0, 0] \quad \text{and} \quad T(1) = [1, 0, -1]$$

Since we are using the standard basis C for $\mathbb{R}^3$, the matrix $\boldsymbol{A}_{BC}$ for T is the matrix whose columns are these images. Thus

$$\boldsymbol{A}_{BC} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

(*b*) To find $\boldsymbol{A}_{DE}$, we make use of the following relationship :

$$\boldsymbol{A}_{DE} = \boldsymbol{Q}\boldsymbol{A}_{BC}\,\boldsymbol{P}^{-1} \qquad \text{...(1)}$$

where $\boldsymbol{P}$ is the transition matrix from B to D and Q is the transition matrix C to E. Since $\boldsymbol{P}$ is the transition matrix from B to D, therefore $\boldsymbol{P}^{-1}$ is the transition matrix from D to B. To compute $\boldsymbol{P}^{-1}$, we need to convert the polynomials in D into vectors in $\mathbb{R}^4$. This is done by converting each polynomial $ax^3 + bx^2 + cx + d$ in D to $[a, b, c, d]$. Thus

$$(x^3 + x^2) \to [1, 1, 0, 0]; \ (x^2 + x) \to [0, 1, 1, 0]; \ (x + 1) \to [0, 0, 1, 1]; \ (1) \to [0, 0, 0, 1]$$

Since B is the standard basis for $\mathbb{R}^3$, the transition matrix ($\boldsymbol{P}^{-1}$) from D to B is obtained by using each of these vectors as columns :

$$\boldsymbol{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

To find $\boldsymbol{Q}$, we first find $\boldsymbol{Q}^{-1}$, the transition matrix from E to C, which is the matrix whose columns are the vectors in E.

$$\boldsymbol{Q}^{-1} = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -3 & -6 \\ -3 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \qquad \boldsymbol{Q} = (\boldsymbol{Q}^{-1})^{-1} = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -3 & -6 \\ -3 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix}$$

Hence, $$\boldsymbol{A}_{DE} = \boldsymbol{Q}\boldsymbol{A}_{BC}\boldsymbol{P}^{-1} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -10 & -15 & -9 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{bmatrix}.$$

6.3 LINEAR OPERATORS AND SIMILARITY

In this section we will show that any two matrices for the same linear operator (on a finite-dimensional vector space) with respect to different ordered bases are similar.

Let V be a finite-dimensional vector space with ordered bases B and C, and let $T : V \to V$ be a linear operator. Then we can find two matrices, $\boldsymbol{A}_{BB}$ and $\boldsymbol{A}_{CC}$, for T with respect to ordered bases B and C, respectively. We will show that $\boldsymbol{A}_{BB}$ and $\boldsymbol{A}_{CC}$ are similar. To prove this, let $\boldsymbol{P}$ denote the transition matrix ($\boldsymbol{P}_{C \leftarrow B}$) from B to C. Then by Theorem 6.6, we have

$$\boldsymbol{A}_{CC} = \boldsymbol{P}\,\boldsymbol{A}_{BB}\,\boldsymbol{P}^{-1} \qquad \Rightarrow \qquad \boldsymbol{A}_{BB} = \boldsymbol{P}^{-1}\,\boldsymbol{A}_{CC}\,\boldsymbol{P}$$

This shows that the matrices $\boldsymbol{A}_{BB}$ and $\boldsymbol{A}_{CC}$ are similar. We have thus proved the following:

THEOREM 6.7 Let V be a finite-dimensional vector space with ordered bases B and C. Let T be a linear operator on V. Then the matrix $\boldsymbol{A}_{BB}$ for T with respect to the basis B is similar to the matrix $\boldsymbol{A}_{CC}$ for T with respect to the basis C. More specifically, if $\boldsymbol{P}$ is the transition matrix from B to C, then $\boldsymbol{A}_{BB} = \boldsymbol{P}^{-1}\,\boldsymbol{A}_{CC}\,\boldsymbol{P}$.