Matrix Multiplication

Suppose we buy 2 CDs at \$3 each and 4 Zip disks at \$5 each. We calculate our total cost by computing the products' price $\times$ quantity and adding:

$$\text{Cost} = 3 \times 2 + 5 \times 4 = \$26$$

Let us instead put the prices in a row vector

$$P = [3 \quad 5] \qquad \text{The price matrix}$$

and the quantities purchased in a column vector,

$$Q = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad \text{The quantity matrix}$$

Q: *Why a row and a column?*

A: It's rather a long story, but mathematicians found that it works best this way . . .

Because P represents the prices of the items we are purchasing and Q represents the quantities, it would be useful if the product PQ represented the total cost, a *single number* (which we can think of as a 1×1 matrix). For this to work, PQ should be calculated the same way we calculated the total cost:

$$PQ = [3 \quad 5]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = [3 \times 2 + 5 \times 4] = [26]$$

Notice that we obtain the answer by multiplying each entry in P (going from left to right) by the corresponding entry in Q (going from top to bottom) and then adding the results. ■

The Product *Row* × *Column*

The **product** AB of a row matrix A and a column matrix B is a 1×1 matrix. The length of the row in A must match the length of the column in B for the product to be defined. To find the product, multiply each entry in A (going from left to right) by the corresponding entry in B (going from top to bottom) and then add the results.

Visualizing

$$[2 \quad 4 \quad 1]\begin{bmatrix} 2 \\ 10 \\ -1 \end{bmatrix} \qquad \begin{array}{rcrl} 2 \times 2 & = & 4 & \text{Product of first entries} = 4 \\ 4 \times 10 & = & 40 & \text{Product of second entries} = 40 \\ 1 \times (-1) & = & \underline{-1} & \text{Product of third entries} = -1 \\ & & 43 & \text{Sum of products} = 43 \end{array}$$

quick Examples

1. $[2 \quad 1]\begin{bmatrix} -3 \\ 1 \end{bmatrix} = [2 \times (-3) + 1 \times 1] = [-6 + 1] = [-5]$

2. $[2 \quad 4 \quad 1]\begin{bmatrix} 2 \\ 10 \\ -1 \end{bmatrix} = [2 \times 2 + 4 \times 10 + 1 \times (-1)] = [4 + 40 + (-1)] = [43]$

Example: Revenue

January sales at the A-Plus auto parts stores in Vancouver and Quebec are given in the following table.

	Vancouver	Quebec
Wiper Blades	20	15
Cleaning Fluid (bottles)	10	12
Floor Mats	8	4

The usual selling prices for these items are \$7.00 each for wiper blades, \$3.00 per bottle for cleaning fluid, and \$12.00 each for floor mats. The discount prices for A-Plus Club members are \$6.00 each for wiper blades, \$2.00 per bottle for cleaning fluid, and \$10.00 each for floor mats. Use matrix multiplication to compute the total revenue at each store, assuming first that all items were sold at the usual prices, and then that they were all sold at the discount prices.

Solution We can do all of the requested calculations at once with a single matrix multiplication. Consider the following two labeled matrices.

$$Q = \begin{array}{r} \\ \textbf{Wb} \\ \textbf{Cf} \\ \textbf{Fm} \end{array} \begin{array}{c} \begin{array}{cc} \textbf{V} & \textbf{Q} \end{array} \\ \begin{bmatrix} 20 & 15 \\ 10 & 12 \\ 8 & 4 \end{bmatrix} \end{array}$$

$$P = \begin{array}{r} \\ \textbf{Usual} \\ \textbf{Discount} \end{array} \begin{array}{c} \begin{array}{ccc} \textbf{Wb} & \textbf{Cf} & \textbf{Fm} \end{array} \\ \begin{bmatrix} 7.00 & 3.00 & 12.00 \\ 6.00 & 2.00 & 10.00 \end{bmatrix} \end{array}$$

The first matrix records the quantities sold, while the second records the sales prices under the two assumptions. To compute the revenue at both stores under the two different assumptions, we calculate $R = PQ$.

$$R = PQ = \begin{bmatrix} 7.00 & 3.00 & 12.00 \\ 6.00 & 2.00 & 10.00 \end{bmatrix} \begin{bmatrix} 20 & 15 \\ 10 & 12 \\ 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 266.00 & 189.00 \\ 220.00 & 154.00 \end{bmatrix}$$

We can label this matrix as follows.

$$R = \begin{array}{r} \\ \textbf{Usual} \\ \textbf{Discount} \end{array} \begin{array}{c} \begin{array}{cc} \textbf{V} & \textbf{Q} \end{array} \\ \begin{bmatrix} 266.00 & 189.00 \\ 220.00 & 154.00 \end{bmatrix} \end{array}$$

In other words, if the items were sold at the usual price, then Vancouver had a revenue of \$266 while Quebec had a revenue of \$189, and so on.

Matrix Inversion

Now that we've discussed matrix addition, subtraction, and multiplication, you may well be wondering about matrix *division*. In the realm of real numbers, division can be thought of as a form of multiplication: Dividing 3 by 7 is the same as multiplying 3 by $1/7$, the inverse of 7. In symbols, $3 \div 7 = 3 \times (1/7)$, or 3×7^{-1}. In order to imitate division of real numbers in the realm of matrices, we need to discuss the multiplicative **inverse,** A^{-1}, of a matrix A.

Note Because multiplication of real numbers is commutative, we can write, for example, $\frac{3}{7}$ as either 3×7^{-1} or $7^{-1} \times 3$. In the realm of matrices, multiplication is not commutative, so from now on we shall *never* talk about "division" of matrices (by "$\frac{B}{A}$" should we mean $A^{-1}B$ or BA^{-1}?). ■

Before we try to find the inverse of a matrix, we must first know exactly what we *mean* by the inverse. Recall that the inverse of a number a is the number, often written a^{-1}, with the property that $a^{-1} \cdot a = a \cdot a^{-1} = 1$. For example, the inverse of 76 is the number $76^{-1} = 1/76$, because $(1/76) \cdot 76 = 76 \cdot (1/76) = 1$. This is the number calculated by the x^{-1} button found on most calculators. Not all numbers have an inverse.

For example—and this is the only example—the number 0 has no inverse, because you cannot get 1 by multiplying 0 by anything.

The inverse of a matrix is defined similarly. To make life easier, we shall restrict attention to **square** matrices, matrices that have the same number of rows as columns.[15]

Inverse of a Matrix

The **inverse** of an $n \times n$ matrix A is that $n \times n$ matrix A^{-1} which, when multiplied by A on either side, yields the $n \times n$ identity matrix I. Thus,

$$AA^{-1} = A^{-1}A = I$$

If A has an inverse, it is said to be **invertible.** Otherwise, it is said to be **singular.**

S **1.** The inverse of the 1×1 matrix $[3]$ is $[1/3]$, because $[3][1/3] = [1] = [1/3][3]$.

2. The inverse of the $n \times n$ identity matrix I is I itself, because $I \times I = I$. Thus, $I^{-1} = I$.

3. The inverse of the 2×2 matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, because

$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad AA^{-1} = I$$

and

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^{-1}A = I$$

Notes

1. It is possible to show that if A and B are square matrices with $AB = I$, then it must also be true that $BA = I$. In other words, once we have checked that $AB = I$, we know that B is the inverse of A. The second check, that $BA = I$, is unnecessary.
2. If B is the inverse of A, then we can also say that A is the inverse of B (why?). Thus, we sometimes refer to such a pair of matrices as an **inverse pair** of matrices. ■

Example **1** Singular Matrix

Can $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have an inverse?

Solution No. To see why not, notice that both entries in the second row of AB will be 0, no matter what B is. So AB cannot equal I, no matter what B is. Hence, A is singular.

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and

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Finding the Inverse of a Square Matrix

Q: In the box, it was stated that the inverse of $\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$ *is* $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$*. How was that obtained?*

A: We can think of the problem of finding A^{-1} as a problem of finding four unknowns, the four unknown entries of A^{-1}:

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

These unknowns must satisfy the equation $AA^{-1} = I$, or

$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we were to try to find the first column of A^{-1}, consisting of x and z, we would have to solve

$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} x - z &= 1 \\ -x - z &= 0 \end{aligned}$$

To solve this system by Gauss-Jordan reduction, we would row-reduce the augmented matrix, which is A with the column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ adjoined.

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & z \end{array}\right]$$

To find the second column of A^{-1} we would similarly row-reduce the augmented matrix obtained by tacking on to A the second column of the identity matrix.

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & y \\ 0 & 1 & w \end{array}\right]$$

The row operations used in doing these two reductions would be exactly the same. We could do both reductions simultaneously by "doubly augmenting" A, putting both columns of the identity matrix to the right of A.

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & x & y \\ 0 & 1 & z & w \end{array}\right]$$

We carry out this reduction in the following example. ■

Example 2 Computing Matrix Inverse

Find the inverse of each matrix.

a. $P = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$ **b.** $Q = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

a. As described above, we put the matrix P on the left and the identity matrix I on the right to get a 2×4 matrix.

$$\underbrace{\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right]}_{P \qquad\qquad I}$$

We now row reduce the whole matrix:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \end{matrix} \to \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} \begin{matrix} 2R_1 - R_2 \\ \\ \end{matrix}$$

$$\to \begin{bmatrix} 2 & 0 & 1 & -1 \\ 0 & -2 & 1 & 1 \end{bmatrix} \begin{matrix} \frac{1}{2}R_1 \\ -\frac{1}{2}R_2 \end{matrix} \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\qquad\qquad\qquad\qquad I \qquad P^{-1}$$

We have now solved the systems of linear equations that define the entries of P^{-1}. Thus,

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

b. The procedure to find the inverse of a 3×3 matrix (or larger) is just the same as for a 2×2 matrix. We place Q on the left and the identity matrix (now 3×3) on the right, and reduce.

$$\overset{Q \qquad\qquad I}{\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right]} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{bmatrix} \begin{matrix} 3R_1 + R_3 \\ R_2 - R_3 \\ \\ \end{matrix} \to$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{bmatrix} \begin{matrix} \frac{1}{3}R_1 \\ -\frac{1}{2}R_2 \\ -\frac{1}{3}R_3 \end{matrix} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$\qquad\qquad\qquad\qquad I \qquad Q^{-1}$$

Thus,

$$Q^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

We have already checked that P^{-1} is the inverse of P. You should also check that Q^{-1} is the inverse of Q.

Input-output Matrix

In this section we look at an application of matrix algebra developed by Wassily Leontief (1906–1999) in the middle of the twentieth century. In 1973, he won the Nobel Prize in Economics for this work. The application involves analyzing national and regional economies by looking at how various parts of the economy interrelate. We'll work out some of the details by looking at a simple scenario.

First, we can think of the economy of a country or a region as being composed of various **sectors,** or groups of one or more industries. Typical sectors are the manufacturing sector, the utilities sector, and the agricultural sector. To introduce the basic concepts, we shall consider two specific sectors: the coal-mining sector (Sector 1) and the electric utilities sector (Sector 2). Both produce a commodity: the coal-mining sector produces coal, and the electric utilities sector produces electricity. We measure these products by their dollar value. By **one unit** of a product, we mean $1 worth of that product.

Here is the scenario.

1. To produce one unit ($1 worth) of coal, assume that the coal-mining sector uses 50¢ worth of coal (to power mining machinery, say) and 10¢ worth of electricity.

2. To produce one unit ($1 worth) of electricity, assume that the electric utilities sector uses 25¢ worth of coal and 25¢ worth of electricity.

These are *internal* usage figures. In addition to this, assume that there is an *external* demand (from the rest of the economy) of 7000 units ($7,000 worth) of coal and 14,000 units ($14,000 worth) of electricity over a specific time period (one year, say). Our basic question is: How much should each of the two sectors supply in order to meet both internal and external demand?

The key to answering this question is to set up equations of the form:

$$\text{Total supply} = \text{Total demand}$$

The unknowns, the values we are seeking, are

$$x_1 = \text{the total supply (in units) from Sector 1 (coal) and}$$
$$x_2 = \text{the total supply (in units) from Sector 2 (electricity)}$$

Our equations then take the following form:

Total supply from Sector 1 = Total demand for Sector 1 products

$$\underset{\text{Coal required by Sector 1}}{x_1 = 0.50x_1} \quad + \quad \underset{\text{Coal required by Sector 2}}{0.25x_2} \quad + \quad \underset{\text{External demand for coal}}{7000}$$

Total supply from Sector 2 = Total demand for Sector 2 products

$$\underset{\text{Electricity required by Sector 1}}{x_2 = 0.10x_1} \quad + \quad \underset{\text{Electricity required by Sector 2}}{0.25x_2} \quad + \quad \underset{\text{External demand for electricity}}{14{,}000}$$

This is a system of two linear equations in two unknowns:

$$x_1 = 0.50x_1 + 0.25x_2 + 7000$$
$$x_2 = 0.10x_1 + 0.25x_2 + 14{,}000$$

We can rewrite this system of equations in matrix form as follows:

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{Production}} = \underbrace{\begin{bmatrix} 0.50 & 0.25 \\ 0.10 & 0.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{Internal demand}} + \underbrace{\begin{bmatrix} 7000 \\ 14{,}000 \end{bmatrix}}_{\text{External demand}}$$

In symbols,

$$X = AX + D$$

Here,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is called the **production vector.** Its entries are the amounts produced by the two sectors. The matrix

$$D = \begin{bmatrix} 7000 \\ 14{,}000 \end{bmatrix}$$

is called the **external demand** vector, and

$$A = \begin{bmatrix} 0.50 & 0.25 \\ 0.10 & 0.25 \end{bmatrix}$$

is called the **technology matrix.** The entries of the technology matrix have the following meanings:

a_{11} = units of Sector 1 needed to produce one unit of Sector 1

a_{12} = units of Sector 1 needed to produce one unit of Sector 2

a_{21} = units of Sector 2 needed to produce one unit of Sector 1

a_{22} = units of Sector 2 needed to produce one unit of Sector 2

You can remember this order by the slogan "In the side, out the top."

Now that we have the matrix equation

$$X = AX + D$$

we can solve it as follows. First, subtract AX from both sides:

$$X - AX = D$$

Because $X = IX$, where I is the 2×2 identity matrix, we can rewrite this as

$$IX - AX = D$$

Now factor out X:

$$(I - A)X = D$$

If we multiply both sides by the inverse of $(I - A)$, we get the solution

$$X = (I - A)^{-1}D$$

Input-Output Model

In an input-output model, an economy (or part of one) is divided into n **sectors.** We then record the $n \times n$ **technology matrix** A, whose ijth entry is the number of units from Sector i used in producing one unit from Sector j ("in the side, out the top"). To meet an **external demand** of D, the economy must produce X, where X is the **production vector.** These are related by the equations

$$X = AX + D$$

or

$$X = (I - A)^{-1}D \qquad \text{Provided } (I - A) \text{ is invertible}$$

Example

In the scenario above, $A = \begin{bmatrix} 0.50 & 0.25 \\ 0.10 & 0.25 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $D = \begin{bmatrix} 7{,}000 \\ 14{,}000 \end{bmatrix}$.

The solution is

$$X = (I - A)^{-1}D$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.50 & 0.25 \\ 0.10 & 0.25 \end{bmatrix} \right)^{-1} \begin{bmatrix} 7{,}000 \\ 14{,}000 \end{bmatrix} \qquad \text{Calculate } I - A$$

$$= \begin{bmatrix} 0.50 & -0.25 \\ -0.10 & 0.75 \end{bmatrix}^{-1} \begin{bmatrix} 7{,}000 \\ 14{,}000 \end{bmatrix} \qquad \text{Calculate } (I - A)^{-1}$$

$$= \begin{bmatrix} \frac{15}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{10}{7} \end{bmatrix} \begin{bmatrix} 7{,}000 \\ 14{,}000 \end{bmatrix}$$

$$= \begin{bmatrix} 25{,}000 \\ 22{,}000 \end{bmatrix}$$

In other words, to meet the demand, the economy must produce \$25,000 worth of coal and \$22,000 worth of electricity.

The next example uses actual data from the U.S. economy (we have rounded the figures to make the computations less complicated). It is rare to find input-output data already packaged for you as a technology matrix. Instead, the data commonly found in statistical sources come in the form of "input-output tables," from which we will have to construct the technology matrix.

Example: Petroleum and Natural Gas

Consider two sectors of the U.S. economy: crude petroleum and natural gas (*crude*) and petroleum refining and related industries (*refining*). According to government figures,* in 1998 the crude sector used $27,000 million worth of its own products and $750 million worth of the products of the refining sector to produce $87,000 million worth of goods (crude oil and natural gas). The refining sector in the same year used $59,000 million worth of the products of the crude sector and $15,000 million worth of its own products to produce $140,000 million worth of goods (refined oil and the like). What was the technology matrix for these two sectors? What was left over from each of these sectors for use by other parts of the economy or for export?

Solution First, for convenience, we record the given data in the form of a table, called the **input-output table.** (All figures are in millions of dollars.)

	To	**Crude**	**Refining**
From	**Crude**	27,000	59,000
	Refining	750	15,000
	Total Output	87,000	140,000

The entries in the top portion are arranged in the same way as those of the technology matrix: The *ij*th entry represents the number of units of Sector *i* that went to Sector *j*. Thus, for instance, the 59,000 million entry in the 1, 2 position represents the number of units of Sector 1, crude, that were used by Sector 2, refining. ("In the side, out the top".)

We now construct the technology matrix. The technology matrix has entries a_{ij} = units of Sector *i* used to produce *one* unit of Sector *j*. Thus,

a_{11} = units of crude to produce one unit of crude. We are told that 27,000 million units of crude were used to produce 87,000 million units of crude. Thus, to produce *one* unit of crude, $27{,}000/87{,}000 \approx 0.31$ units of crude were used, and so $a_{11} \approx 0.31$. (We have rounded this value to two significant digits; further digits are not reliable due to rounding of the original data.)

a_{12} = units of crude to produce one unit of refined:
$$a_{12} = 59{,}000/140{,}000 \approx 0.42$$

a_{21} = units of refined to produce one unit of crude:
$$a_{21} = 750/87{,}000 \approx 0.0086$$

a_{22} = units of refined to produce one unit of refined:
$$a_{22} = 15{,}000/140{,}000 \approx 0.11$$

This gives the technology matrix

$$A = \begin{bmatrix} 0.31 & 0.42 \\ 0.0086 & 0.11 \end{bmatrix}$$ Technology Matrix

In short *we obtained the technology matrix from the input-output table by dividing the Sector 1 column by the Sector 1 total, and the Sector 2 column by the Sector 2 total.*

Now we also know the total output from each sector, so *we have already been given the production vector:*

$$X = \begin{bmatrix} 87{,}000 \\ 140{,}000 \end{bmatrix} \qquad \text{Production Vector}$$

What we are asked for is the external demand vector D, the amount available for the outside economy. To find D, we use the equation

$$X = AX + D \qquad \text{Relationship of } X,\ A,\ \text{and } D$$

where, this time, we are given A and X, and must solve for D. Solving for D gives

$$\begin{aligned} D &= X - AX \\ &= \begin{bmatrix} 87{,}000 \\ 140{,}000 \end{bmatrix} - \begin{bmatrix} 0.31 & 0.42 \\ 0.0086 & 0.11 \end{bmatrix} \begin{bmatrix} 87{,}000 \\ 140{,}000 \end{bmatrix} \\ &\approx \begin{bmatrix} 87{,}000 \\ 140{,}000 \end{bmatrix} - \begin{bmatrix} 86{,}000 \\ 16{,}000 \end{bmatrix} = \begin{bmatrix} 1000 \\ 124{,}000 \end{bmatrix} \qquad \text{We rounded to 2 digits}^{\dagger} \end{aligned}$$

The first number, \$1000 million, is the amount produced by the crude sector that is available to be used by other parts of the economy or to be exported. (In fact, because something has to happen to all that crude petroleum and natural gas, this is the amount actually used or exported, where use can include stockpiling.) The second number, \$124,000 million, represents the amount produced by the refining sector that is available to be used by other parts of the economy or to be exported. Complete the example using a TI-83/84 or Excel.

Note that we could have calculated D more simply from the input-output table. The internal use of units from the crude sector was the sum of the outputs from that sector:

$$27{,}000 + 59{,}000 = 86{,}000$$