CHAPTER

6

Linear Transformations

LEARNING OBJECTIVES

After studying the material in this chapter, you should be able to :

- ***Know a special class of functions, known as linear transformations.***
- ***Understand elementary properties of linear transformations.***
- ***Find a linear transformation by knowing its action an a basis.***
- ***Find the matrix of a linear transformation.***
- ***Know the Dimension Theorem, exhibiting an important relationship between the dimensions of the domain and the range of a linear transformation.***
- ***Identify two special types of linear transformations : one-to-one and onto.***
- ***Determine whether two vector spaces are isomorphic.***
- ***Represent each two-dimensional point by a corresponding set of homogeneous coordinates.***
- ***Represent all possible movements using matrix multiplication in homogeneous coordinates.***
- ***Use Similarity Method to perform movements that are not centered at the origin.***
- ***Find the matrix for any composition of translations, rotations, reflections and scaling.***

6.1 INTRODUCTION TO LINEAR TRANSFORMATIONS

In this section we introduce a special class of functions, known as linear transformations, that map vectors in one vector space to those in another. We will also examine some elementary properties of linear transformations.

DEFINITION Linear Transformation

Let V and W be two vector spaces over $\mathbb{R}$. A function

$$T : V \to W$$

is called a **linear transformation** from V to W if it satisfies the following properties:

1. $T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)$, for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$

2. $T(\alpha\, \boldsymbol{v}) = \alpha\, T(\boldsymbol{v})$, for all $\alpha \in \mathbb{R}$ and all $\boldsymbol{v} \in V$.

Thus, a linear transformation is a function from one vector space to another that preserves the operations of addition and scalar multiplication.

Note Notice that the two conditions for linearity are equivalent to a single condition

$$T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) = \alpha T(\boldsymbol{v}_1) + \beta T(\boldsymbol{v}_2), \text{ for all } \boldsymbol{v}_1, \boldsymbol{v}_2 \in V \text{ and all } \alpha, \beta \in \mathbb{R}.$$

EXAMPLE 1 Zero Linear Transformation

Let V and W be vector spaces. Consider the mapping $T : V \to W$ defined by $T(\boldsymbol{v}) = \mathbf{0}_W$, for all $\boldsymbol{v} \in V$.

We will show that T is a linear transformation.

1. We must show that $T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)$, for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$

Now $T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \mathbf{0}_W = \mathbf{0}_W + \mathbf{0}_W = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)$.

2. We must show that $T(\alpha \boldsymbol{v}) = \alpha T(\boldsymbol{v})$, for all $\alpha \in \mathbb{R}$ and for all $\boldsymbol{v} \in V$

Now $T(\alpha \boldsymbol{v}) = \mathbf{0}_W = \alpha \mathbf{0}_W = \alpha T(\boldsymbol{v})$.

Hence, T is a linear transformation, known as the **zero linear transformation**.

EXAMPLE 2 Let

$$V = \mathcal{M}_{mn}, \text{ the space of all } m \times n \text{ matrices}$$

and

$$W = \mathcal{M}_{nm}, \text{ the space of all } n \times m \text{ matrices}$$

Consider the mapping $T : V \to W$ defined by

$$T(\boldsymbol{A}) = \boldsymbol{A}^{\boldsymbol{T}} \text{ for all } \boldsymbol{A} \in \boldsymbol{V}$$

Show that T is a linear transformation.

SOLUTION Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be any two matrices in $V = \mathcal{M}_{mn}$.

Then

$$T(\boldsymbol{A}_1 + \boldsymbol{A}_2) = (\boldsymbol{A}_1 + \boldsymbol{A}_2)^{\boldsymbol{T}} = \boldsymbol{A}_1^{\boldsymbol{T}} + \boldsymbol{A}_2^{\boldsymbol{T}} = T(\boldsymbol{A}_1) + T(\boldsymbol{A}_2)$$

Similarly,

$$T(\alpha \boldsymbol{A}) = (\alpha \boldsymbol{A})^{\boldsymbol{T}} = \alpha \boldsymbol{A}^{\boldsymbol{T}} = \alpha T(\boldsymbol{A}), \text{ for any } \alpha \in \mathbb{R} \text{ and } \boldsymbol{A} \in \boldsymbol{V}$$

Hence, T is a linear transformation from $\mathcal{M}_{mn}$ to $\mathcal{M}_{nm}$.

EXAMPLE 3 Let

$$V = \mathcal{P}_n, \text{ the space of all polynomials of degree} \le n, \text{ with real coefficients}$$

and

$$W = \mathcal{P}_{n-1}, \text{ the space of all polynomials of degree} \le n-1, \text{ with real coefficients}$$

Consider the mapping $T : V \to W$ defined by

$$T(\boldsymbol{p}) = \boldsymbol{p}' \text{ for any } \boldsymbol{p} \in \boldsymbol{V} = \mathcal{P}_n$$

Show that T is a linear transformation.

SOLUTION For any $\boldsymbol{p}_1, \boldsymbol{p}_2 \in V$, we have

$$T(\boldsymbol{p}_1 + \boldsymbol{p}_2) = (\boldsymbol{p}_1 + \boldsymbol{p}_2)' = \boldsymbol{p}_1' + \boldsymbol{p}_2' = T(\boldsymbol{p}_1) + T(\boldsymbol{p}_2)$$

Similarly,

$$T(\alpha \boldsymbol{p}) = (\alpha \boldsymbol{p})' = \alpha \boldsymbol{p}' = \alpha T(\boldsymbol{p}), \text{ for any } \alpha \in \mathbb{R} \text{ and } \boldsymbol{p} \in \boldsymbol{V}$$

Hence, T is a linear transformation.

EXAMPLE 4 Let V be an n-dimensional vector space over $\mathbb{R}$, and let B be an ordered basis for V. Then every vector $\boldsymbol{v} \in V$ has its coordinatization $[\boldsymbol{v}]_B$ with respect to B satisfying the following properties

$$[\boldsymbol{v}_1 + \boldsymbol{v}_2]_B = [\boldsymbol{v}_1]_B + [\boldsymbol{v}_2]_B, \quad \text{for all } \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$$

$$[\alpha \boldsymbol{v}]_B = \alpha [\boldsymbol{v}]_B, \quad \text{for all } \alpha \in \mathbb{R}, \text{ and for all } \boldsymbol{v} \in V$$

Consider the mapping $T : V \to \mathbb{R}^n$ defined by

$$T(\boldsymbol{v}) = [\boldsymbol{v}]_B \quad \text{for any } \boldsymbol{v} \in V$$

We will show that T is a linear transformation. Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be any two vectors in V. Then from the properties of coordinatization just stated, we have

$$T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = [\boldsymbol{v}_1 + \boldsymbol{v}_2]_B = [\boldsymbol{v}_1]_B + [\boldsymbol{v}_2]_B = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)$$

Similarly,

$$T(\alpha \boldsymbol{v}) = [\alpha \boldsymbol{v}]_B = \alpha [\boldsymbol{v}]_B = \alpha T(\boldsymbol{v}), \quad \text{for any } \alpha \in \mathbb{R} \text{ and } \boldsymbol{v} \in V$$

Hence, T is a linear transformation.

DEFINITION Linear Operator

Let V be a vector space. A linear transformation $T : V \to V$ is called a **linear operator**. Thus, a linear operator is a linear transformation from a vector space to itself.

EXAMPLE 5 Identity Linear Operator

Let V be a vector space. Consider the mapping $T : V \to V$ defined by $T(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$.

We will show that T is a linear operator. Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$. Then

$$T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \boldsymbol{v}_1 + \boldsymbol{v}_2 = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)$$

Also, let $\boldsymbol{v} \in V$ and $\alpha \in \mathbb{R}$. Then

$$T(\alpha \boldsymbol{v}) = \alpha \boldsymbol{v} = \alpha T(\boldsymbol{v})$$

Hence, T is a linear operator, known as the **Identity Linear Operator.**

EXAMPLE 6 Contractions and Dilations

Let $k \in \mathbb{R}$. Define $T : \mathbb{R}^n \to \mathbb{R}^n$ as $T(\boldsymbol{v}) = k\boldsymbol{v}$, *for all* $\boldsymbol{v} \in \mathbb{R}^n$.

We will show that T is a linear operator.

1. Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^n$. Then

$$T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = k(\boldsymbol{v}_1 + \boldsymbol{v}_2) = k\boldsymbol{v}_1 + k\boldsymbol{v}_2 = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)$$

2. Let $\boldsymbol{v} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then

$$T(\alpha \boldsymbol{v}) = k(\alpha \boldsymbol{v}) = \alpha(k\boldsymbol{v}) = \alpha T(\boldsymbol{v})$$

Hence, T is a linear operator, called **dilation** or **contraction,** according as $|k| > 1$ or $|k| < 1$, respectively. If $|k| > 1$, T **dilates** (stretches) the length of the vector, and if $|k| < 1$, T **contracts** (shrinks) the length.

EXAMPLE 7 Projections

Consider the mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T([x_1, x_2, x_3]) = [x_1, x_2, 0], \quad [x_1, x_2, x_3] \in \mathbb{R}^3$$

We will show that T is a linear operator.

1. Let $\boldsymbol{v_1} = [x_1, x_2, x_3]$, $\boldsymbol{v_2} = [y_1, y_2, y_3] \in \mathbb{R}^3$. Then

$$\begin{aligned} T(\boldsymbol{v_1} + \boldsymbol{v_2}) &= T([x_1, x_2, x_3] + [y_1, y_2, y_3]) \\ &= T([x_1 + y_1, x_2 + y_2, x_3 + y_3]) \\ &= [x_1 + y_1, x_2 + y_2, 0] \\ &= [x_1, x_2, 0] + [y_1, y_2, 0] \\ &= T([x_1, x_2, x_3]) + T([y_1, y_2, y_3]) \\ &= T(\boldsymbol{v_1}) + T(\boldsymbol{v_2}) \end{aligned}$$

2. Let $\boldsymbol{v} = [x_1, x_2, x_3] \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T(\alpha\, \boldsymbol{v}) &= T(\alpha[x_1, x_2, x_3]) \\ &= T([\alpha x_1, \alpha x_2, \alpha x_3]) \\ &= [\alpha x_1, \alpha x_2, 0] \\ &= \alpha[x_1, x_2, 0] \\ &= \alpha T([x_1, x_2, x_3]) \\ &= \alpha T(\boldsymbol{v}) \end{aligned}$$

Hence, T is a linear operator on $\mathbb{R}^3$, known as a **projection operator**, because of its geometrical interpretation. *It projects each vector in* $\mathbb{R}^3$ *to a corresponding vector in the xy-plane* (see Fig. 6.1).

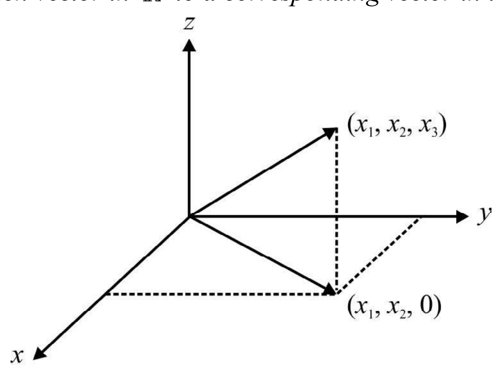

FIGURE 6.1

Note Notice that we can also define a projection operator on $\mathbb{R}^3$ which projects each vector in $\mathbb{R}^3$ to a corresponding vector in the *yz*-plane or the *zx*-plane.

EXAMPLE 8 Reflections

Consider the mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T([x_1, x_2, x_3]) = [x_1, x_2, -x_3], \quad [x_1, x_2, x_3] \in \mathbb{R}^3$$

We will show that T is a linear operator.

1. Let $\boldsymbol{v_1} = [x_1, x_2, x_3]$, $\boldsymbol{v_2} = [y_1, y_2, y_3] \in \mathbb{R}^3$. Then

$$\begin{aligned} T(\boldsymbol{v_1} + \boldsymbol{v_2}) &= T([x_1, x_2, x_3] + [y_1, y_2, y_3]) \\ &= T([x_1 + y_1, x_2 + y_2, x_3 + y_3]) \\ &= [x_1 + y_1, x_2 + y_2, -(x_3 + y_3)] \\ &= [x_1, x_2, -x_3] + [y_1, y_2, -y_3] \\ &= T([x_1, x_2, x_3]) + T([y_1, y_2, y_3]) \\ &= T(\boldsymbol{v_1}) + T(\boldsymbol{v_2}) \end{aligned}$$

2. Let $\boldsymbol{v} = [x_1, x_2, x_3] \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T(\alpha \boldsymbol{v}) &= T(\alpha [x_1, x_2, x_3]) \\ &= T([\alpha x_1, \alpha x_2, \alpha x_3]) \\ &= [\alpha x_1, \alpha x_2, -\alpha x_3] \\ &= \alpha [x_1, x_2, -x_3] \\ &= \alpha T([x_1, x_2, x_3]) \\ &= \alpha T(\boldsymbol{v}) \end{aligned}$$

Hence, T is a linear operator on $\mathbb{R}^3$, called a **reflection operator**. This operator reflects each vector $[x_1, x_2, x_3]$ through the xy-plane, which acts like a mirror (see Fig. 6.2).

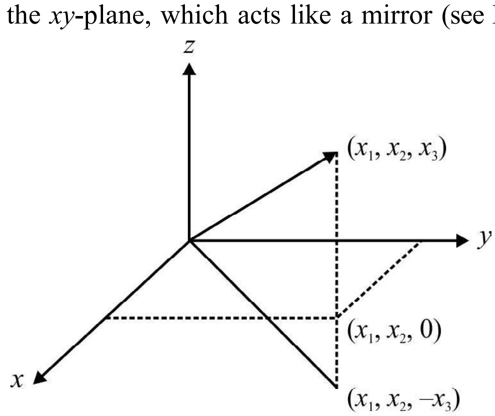

FIGURE 6.2

Note Notice that we can also define a reflection operator on $\mathbb{R}^3$ which reflects each vector in $\mathbb{R}^3$ through the yz-plane or the zx-plane.

EXAMPLE 9 Rotation Linear Operator

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(\boldsymbol{v}) = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix},$$

for all $\boldsymbol{v} = [x, y] \in \mathbb{R}^2$, where θ is fixed angle. We will show that T is a linear operator.

Let $\boldsymbol{v_1} = [x_1, y_1]$, $\boldsymbol{v_2} = [x_2, y_2]$ be two vectors in $\mathbb{R}^2$. Then

$$T(\boldsymbol{v_1} + \boldsymbol{v_2}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} (\boldsymbol{v}_1 + \boldsymbol{v}_2)$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} (\boldsymbol{v}_1) + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} (\boldsymbol{v}_2)$$

$$= T(\boldsymbol{v_1}) + T(\boldsymbol{v_2})$$

Next, let $\alpha \in \mathbb{R}$ and $\boldsymbol{v} = [x, y] \in \mathbb{R}^2$. Then

$$T(\alpha\, \boldsymbol{v}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} (\alpha\, \boldsymbol{v}) = \alpha \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \boldsymbol{v} = \alpha\, T(\boldsymbol{v})$$

Hence T is a linear operator.

Note Notice that the linear operator T defined above rotates the vector $[x, y]$ counterclockwise through the angle θ in the plane (see Figure 6.3). To prove this, consider the vector $[x', y']$ obtained by rotating $[x, y]$ counterclockwise through the angle θ. We can write $x = r\cos\alpha$, $y = r\sin\alpha$, where $r = \sqrt{x^2 + y^2}$, and α is the angle shown in Figure 6.3.

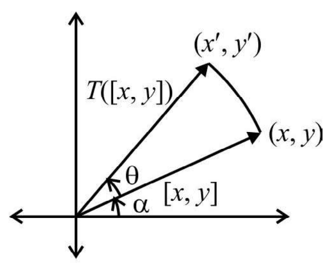

FIGURE 6.3

Also, $x' = r\cos(\theta + \alpha)$, and $y' = r\sin(\theta + \alpha)$

Using the following trigonometry formulas:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

we see that

$$x' = r\cos(\theta + \alpha) = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha = x\cos\theta - y\sin\theta$$

$$y' = r\sin(\theta + \alpha) = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha = x\sin\theta + y\cos\theta$$

$$\therefore \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

i.e., $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

EXAMPLE 10 Let $\boldsymbol{A}$ be a fixed $m \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\boldsymbol{x}) = \boldsymbol{Ax}, \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n$$

Show that T is a linear transformation.

SOLUTION Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$. Then $T(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \boldsymbol{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \boldsymbol{A}\boldsymbol{x}_1 + \boldsymbol{A}\boldsymbol{x}_2 = T\boldsymbol{x}_1 + T\boldsymbol{x}_2$

Also, let $\boldsymbol{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then $T(c\boldsymbol{x}) = \boldsymbol{A}(c\boldsymbol{x}) = c(\boldsymbol{A}\boldsymbol{x}) = cT(\boldsymbol{x})$.

Hence, T is a linear transformation.

EXAMPLE 11 Shear Operators

Let k be a fixed scalar in $\mathbb{R}$. Consider the function $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

Show that T is a linear operator. **[*Delhi Univ. GE*-2, 2018]**

SOLUTION Let $\boldsymbol{v_1} = [x_1, y_1]$ and $\boldsymbol{v_2} = [x_2, y_2]$ be two vectors in $\mathbb{R}^2$. Then

$$T(\boldsymbol{v_1} + \boldsymbol{v_2}) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}(\boldsymbol{v_1} + \boldsymbol{v_2}) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\boldsymbol{v}_1 + \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\boldsymbol{v}_2 = T(\boldsymbol{v_1}) + T(\boldsymbol{v_2})$$

Next, let $\alpha \in \mathbb{R}$ and $\boldsymbol{v} = [x, y] \in \mathbb{R}^2$. Then

$$T(\alpha\, \boldsymbol{v}) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}(\alpha \boldsymbol{v}) = \alpha\left(\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\boldsymbol{v}\right) = \alpha\, T(\boldsymbol{v})$$

Hence, T is a linear operator, called a **shear in the *x*-axis with factor *k*.**

Similarly, the function $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$

is also a linear operator on $\mathbb{R}^2$, called a **shear in the *y*-direction with factor *k*.**

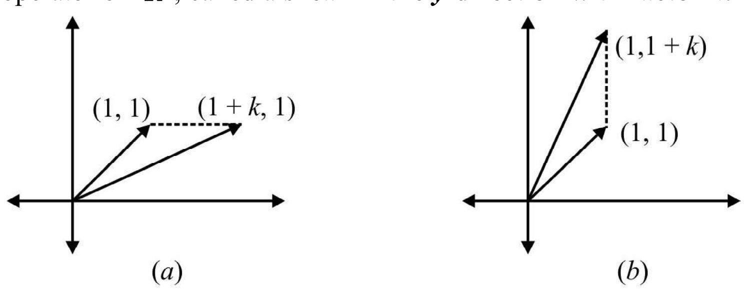

FIGURE 6.4

The following theorem contains some basic properties of linear transformations.

THEOREM 6.1 Properties of Linear Transformations

Let V and W be two vector spaces, and let $T: V \to W$ be a linear transformation. Let $\mathbf{0}_V$ be the zero vector in V and $\mathbf{0}_W$ be the zero vector in W. Then

1. $T(\mathbf{0}_V) = \mathbf{0}_W$
2. $T(-\boldsymbol{v}) = -T(\boldsymbol{v})$, for all $\boldsymbol{v} \in V$
3. $T(\boldsymbol{u} - \boldsymbol{v}) = T(\boldsymbol{u}) - T(\boldsymbol{v})$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$
4. $T(a_1 \boldsymbol{v_1} + a_2 \boldsymbol{v_2} + \ldots + a_n \boldsymbol{v_n}) = a_1 T(\boldsymbol{v_1}) + a_2 T(\boldsymbol{v_2}) + \ldots + a_n T(\boldsymbol{v_n})$, for all $a_1, a_2, \ldots, a_n \in \mathbb{R}$, $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n} \in V$, for $n \geq 1$

Proof 1.

$$\begin{aligned} T(\mathbf{0}_V) &= T(0\ \mathbf{0}_V) \\ &= 0\ T(\mathbf{0}_V) && \text{(Property (2) of linear transformation)} \\ &= \mathbf{0}_W \end{aligned}$$

So, (1) is proved.

2.

$$\begin{aligned} T(-\boldsymbol{v}) &= T((-1)\boldsymbol{v}) \\ &= (-1)T(\boldsymbol{v}) && \text{(Property (2) of linear transformation)} \\ &= -T(\boldsymbol{v}) \end{aligned}$$

So, (2) is proved.

3.

$$\begin{aligned} T(\boldsymbol{u} - \boldsymbol{v}) &= T(\boldsymbol{u} + (-1)\boldsymbol{v}) \\ &= T(\boldsymbol{u}) + T(-1)\boldsymbol{v}) && \text{(Property (1) of linear transformation)} \\ &= T(\boldsymbol{u}) - T(\boldsymbol{v}) && \text{(by part (2))} \end{aligned}$$

4. To prove (4), we use induction on n.

For $n = 1$, we have

$$T(a_1 \boldsymbol{v_1}) = a_1 T(\boldsymbol{v_1}) \qquad \text{(Property (2) of linear transformation)}$$

so, result is true for $n = 1$

Similarly, for $n = 2$, we have

$$\begin{aligned} T(a_1 \boldsymbol{v_1} + a_2 \boldsymbol{v_2}) &= T(a_1 \boldsymbol{v_1}) + T(a_2 \boldsymbol{v_2}) && \text{(Property (1) of linear transformation)} \\ &= a_1 T(\boldsymbol{v_1}) + a_2 T(\boldsymbol{v_2}) && \text{(Property (2) of linear transformation)} \end{aligned}$$

so, the result is also true for $n = 2$

Now, we assume that the result is true for $n = m$, *i.e.,*

$$T(a_1 \boldsymbol{v_1} + a_2 \boldsymbol{v_2} + \ldots + a_m \boldsymbol{v_m}) = a_1 T(\boldsymbol{v_1}) + a_2 T(\boldsymbol{v_2}) + \ldots + a_m T(\boldsymbol{v_m})$$

We have to deduce that the result is also true for $n = m + 1$

We have

$$\begin{aligned} & T(a_1 \boldsymbol{v_1} + a_2 \boldsymbol{v_2} + \ldots + a_m \boldsymbol{v_m} + a_{m+1} \boldsymbol{v_{m+1}}) \\ &= T(a_1 \boldsymbol{v_1} + a_2 \boldsymbol{v_2} + \ldots + a_m \boldsymbol{v_m}) + \mathrm{T}(a_{m+1} \boldsymbol{v_{m+1}}) \\ & \qquad \text{(Property (2) of linear transformation)} \\ &= a_1 T(\boldsymbol{v_1}) + a_2 T(\boldsymbol{v_2}) + \ldots + a_m T(\boldsymbol{v_m}) + a_{m+1} T(\boldsymbol{v_{m+1}}) \\ & \qquad \text{(by the induction hypothesis)} \end{aligned}$$

So, the result is true for $n = m + 1$. Hence, by the principle of mathematical induction, the result is true for any natural number n.

Note Part (1) of Theorem 6.1 can be used to prove that a function is not a linear transformation.

EXAMPLE 12 Let V be a vector space, and let $\boldsymbol{x} \neq \mathbf{0}$ be a fixed vector in V. Prove that the translation function $f : V \to V$ defined by $f(\boldsymbol{v}) = \boldsymbol{v} + \boldsymbol{x}$ is not a linear transformation.

[*Delhi Univ. GE*-2, 2018]

SOLUTION We have

$$f(\mathbf{0}) = \mathbf{0} + \boldsymbol{x} = \boldsymbol{x} \neq \mathbf{0}$$

So, by part (1) of Theorem 6.1 f is not a linear transformation.

Composition of Linear Transformations

If $f : X \to Y$ and $g : Y \to Z$ are functions, then the **composition** of f and g is defined to be the function $gof : X \to Z$ given by $(gof)(x) = g(f(x))$. The following theorem asserts that the composition of linear transformations is again a linear transformation.

THEOREM 6.2 Let V, W and X be vector spaces. Let $T_1 : V \to W$ and $T_2 : W \to X$ be linear transformations. Then the composition function $T_2 o T_1 : V \to X$ given by $(T_2 o T_1)(\boldsymbol{v}) = T_2(T_1(\boldsymbol{v}))$, for all $\boldsymbol{v} \in V$, is a linear transformation.

Proof To show that $T_2 o T_1$ is a linear transformation, we must show that both of the following are true:

$$(T_2 o T_1)(\boldsymbol{v_1} + \boldsymbol{v_2}) = (T_2 o T_1)(\boldsymbol{v_1}) + (T_2 o T_1)(\boldsymbol{v_2}), \quad \text{for all } \boldsymbol{v_1}, \boldsymbol{v_2} \in V$$

$$(T_2 o T_1)(\alpha \boldsymbol{v}) = \alpha(T_2 o T_1)(\boldsymbol{v}), \quad \text{for all } \alpha \in \mathbb{R} \text{ and } \boldsymbol{v} \in V$$

To prove the first property, consider

$$\begin{aligned}
(T_2 o T_1)(\boldsymbol{v_1} + \boldsymbol{v_2}) &= T_2\,(T_1(\boldsymbol{v_1} + \boldsymbol{v_2})) && \text{(definition of composition)}\\
&= T_2\,[T_1(\boldsymbol{v_1}) + T_1(\boldsymbol{v_2})] && (\because\ T_1 \text{ is a Linear Transformation})\\
&= T_2\,(T_1(\boldsymbol{v_1})) + T_2(T_1(\boldsymbol{v_2})) && (\because\ T_2 \text{ is a Linear Transformation})\\
&= (T_2 o T_1)(\boldsymbol{v_1}) + (T_2 o T_1)(\boldsymbol{v_2}) && \text{(definition of composition)}
\end{aligned}$$

So, the first property holds.

To prove the second property, consider

$$\begin{aligned}
(T_2 o T_1)(\alpha \boldsymbol{v}) &= T_2(T_1(\alpha \boldsymbol{v})) && \text{(definition of composition)}\\
&= T_2(\alpha T_1(\boldsymbol{v})) && (\because\ T_1 \text{ is a Linear Transformation})\\
&= \alpha(T_2(T_1(\boldsymbol{v})) && (\because\ T_2 \text{ is a Linear Transformation})\\
&= \alpha(T_2 o T_1)(\boldsymbol{v}) && \text{(definition of composition)}
\end{aligned}$$

So, the second property also holds.

Hence, $T_2 o T_1$ is a linear transformation.

EXAMPLE 13 Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator representing the counterclockwise rotation in $\mathbb{R}^2$ through a fixed angle θ. That is,

$$T_1(\boldsymbol{v}) = T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix},$$

where $\boldsymbol{v} = [x, y] \in \mathbb{R}^2$. Further, let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator representing the reflection of vectors in $\mathbb{R}^2$ through the x-axis. That is,

$$T_2(\boldsymbol{v}) = T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}, \qquad \boldsymbol{v} = [x, y] \in \mathbb{R}^2$$

Because T_1 and T_2 are both linear transformations, Theorem 6.2 asserts that the composition $T_2 o T_1$ of T_1 and T_2 given by

$$(T_2 o T_1)(\boldsymbol{v}) = T_2\left(T_1(\boldsymbol{v})\right) = T_2\left(\begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}\right) = \begin{bmatrix} x\cos\theta - y\sin\theta \\ -x\sin\theta - y\cos\theta \end{bmatrix}$$

is also a linear transformation. Notice that $T_2 o T_1$ represents a counterclockwise rotation of $[x, y]$ through the angle θ followed by a reflection through the x-axis (see Fig. 6.5).

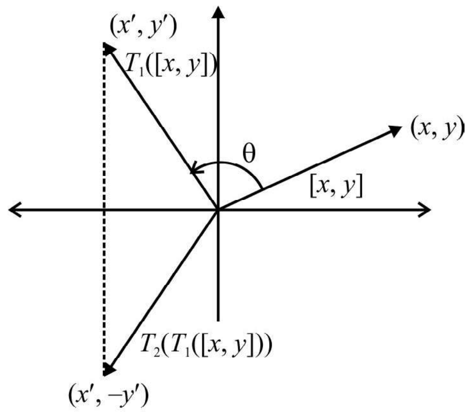

FIGURE 6.5

Linear Transformations and Subspaces

We conclude this section by proving that, under a linear transformation $T: V \to W$, "subspaces" of V are mapped to "subspaces" of W, and vice-versa.

Let V_1 and V_2 be vector spaces and let $T: V_1 \to V_2$ be a linear transformation. Given a set $U \subseteq V_1$, the **image** of U in V_2 is defined to be the set

$$T(U) = \{T(\boldsymbol{u}) : \boldsymbol{u} \in U\}$$

Similarly, given a set $W \subseteq V_2$, the **pre-image** of W in V_1 is defined to be the set

$$T^{-1}(W) = \{\boldsymbol{v} \in V_1 : T(\boldsymbol{v}) \in W\}$$

THEOREM 6.3 Let V_1 and V_2 be vector spaces, and let $T: V_1 \to V_2$ be a linear transformation.

1. If U is a subspace of V_1, then $T(U)$, the image of U in V_2, is a subspace of V_2.

2. If W is a subspace of V_2, then $T^{-1}(W)$, the pre-image of W in V_1, is a subspace of V_1.

Proof **1.** Since U is a subspace of V_1, $\mathbf{0}_{V_1} \in U$. By part (1) of Theorem 6.1 we have

$$\mathbf{0}_{V_2} = T(\mathbf{0}_{V_1}) \in T(U)$$

Thus, $T(U)$ is non-empty. Hence, to show that $T(U)$ is a subspace of V_2, we must show that $T(U)$ is closed under addition and scalar multiplication.

First, suppose that $\boldsymbol{w}_1, \boldsymbol{w}_2$ are any two vectors in $T(U)$. Then, by definition of $T(U)$, we have

$$\boldsymbol{w}_1 = T(\boldsymbol{u}_1) \quad \text{and} \quad \boldsymbol{w}_2 = T(\boldsymbol{u}_2)$$

for some $\boldsymbol{u}_1, \boldsymbol{u}_2 \in U$. So,

$$\boldsymbol{w}_1 + \boldsymbol{w}_2 = T(\boldsymbol{u}_1) + T(\boldsymbol{u}_2) = T(\boldsymbol{u}_1 + \boldsymbol{u}_2) \qquad (\because T \text{ is a L.T.})$$

However, since U is a subspace of V_1, $\boldsymbol{u_1} + \boldsymbol{u_2} \in U$. Thus, $\boldsymbol{w_1} + \boldsymbol{w_2} \in T(U)$. Hence, $T(U)$ is closed under addition.

Next, let $\boldsymbol{w}$ be any vector in $T(U)$, and let α be a scalar. We must show that $\alpha\,\boldsymbol{w} \in T(U)$. By definition of $T(U)$, $\boldsymbol{w} = T(\boldsymbol{u})$, for some $\boldsymbol{u} \in U$. Then

$$\alpha\,\boldsymbol{w} = \alpha T(\boldsymbol{u}) = T(\alpha \boldsymbol{u}) \qquad (\because\ T \text{ is a L.T.})$$

However, since U is a subspace of V_1, $\alpha\,\boldsymbol{u} \in U$, and hence $\alpha\,\boldsymbol{w} \in T(U)$. Thus, $T(U)$ is closed under scalar multiplication.

2. The pre-image of a subspace W of V_2 is given by $T^{-1}(W) = \{\boldsymbol{v} \in V_1 : T(\boldsymbol{v}) \in W\}$

$\because$ $\qquad T(\boldsymbol{0}_{V_1}) = \boldsymbol{0}_{V_2} \in W$, so $\boldsymbol{0}_{V_1} \in T^{-1}(W)$ $\quad\therefore\quad T^{-1}(W)$ is non-empty.

Also, let $\boldsymbol{v_1}, \boldsymbol{v_2} \in T^{-1}(W) \Rightarrow T(\boldsymbol{v_1}), T(\boldsymbol{v_2}) \in W \Rightarrow T(\boldsymbol{v_1}) + T(\boldsymbol{v_2}) \in W \Rightarrow T(\boldsymbol{v_1} + \boldsymbol{v_2}) \in W$

$\Rightarrow \qquad \boldsymbol{v_1} + \boldsymbol{v_2} \in T^{-1}(W)$

Finally, let $\boldsymbol{v} \in T^{-1}(W)$, and let $\alpha \in \mathbb{R}$. Then

$$\boldsymbol{v} \in T^{-1}(W) \Rightarrow T(\boldsymbol{v}) \in W \Rightarrow \alpha\, T(\boldsymbol{v}) \in W \Rightarrow T(\alpha\,\boldsymbol{v}) \in W \Rightarrow \alpha\,\boldsymbol{v} \in T^{-1}(W)$$

Hence $T^{-1}(W)$ is a subspace of V_1.

EXERCISE 6.1

1. Determine which of the following functions are linear transformations.

(*a*) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T([x, y]) = [2x - 3y, 3x + 4y]$

(*b*) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T([x_1, x_2, x_3]) = [x_1 + 1, x_2 - 2, x_3] = [x_1, x_2, x_3] + [1, -2, 0]$

(*c*) $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T([x, y]) = \sqrt{x^2 + y^2}$.

(*d*) $T : \mathbb{R}^4 \to \mathbb{R}$ given by $T([x_1, x_2, x_3, x_4]) = |x_1|$

(*e*) $T : \mathcal{P}_2 \to \mathbb{R}$ given by $T(a_2 x^2 + a_1 x + a_0) = a_2 + a_1 + a_0$

(*f*) $T : \mathcal{M}_{22} \to \mathbb{R}$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(*g*) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T([x_1, x_2, x_3]) = [e^{x_1}, \cos x_2, \sin x_3]$

2. Show that the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T([x_1, x_2, x_3]) = [-x_1, x_2, x_3]$ is a linear operator.

3. Let $\boldsymbol{x}$ be a fixed vector in $\mathbb{R}^n$. Prove that the mapping $T : \mathbb{R}^n \to \mathbb{R}$ given by $T(\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y}$ is a linear transformation.

4. (*a*) Show that the mapping $T : \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $T(\boldsymbol{A}) = \boldsymbol{A} + \boldsymbol{A^T}$ is a linear operator on $\mathcal{M}_{nn}$.