Supplementary Study Material
GE – IV Mathematics (Semester – IV)
Elements of Analysis

Unit – III : Power Series

Contents:

Element of Analysis
Semester – IV (GE – IV Maths)
Chapter 5 – Power Series

Definition: Power Series (P.S.):

An infinite series of the form:

$$\boxed{\sum_{n=0}^{\infty} a_n \cdot x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots}$$

where a_i's are real constants & $x \in \mathbb{R}$ is a real variable is called a power series centred at 0.

ex's (1) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ (P.S. having centre = 0)

(2) $\sum_{n=0}^{\infty} n! \cdot x^n = 1 + x + 2!\, x^2 + 3!\, x^3 + \cdots$ (P.S having centre 0)

Note: More Generally, an infinite series of the form:

$$\boxed{\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots}$$

is called a power series centred at 'c'.

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ex's (1) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} = 1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \cdots$

is a power series having centre $C = 2$.

(2) $\sum_{n=0}^{\infty} (x+5)^n \cdot n! = 1 + (x+5) + (x+5)^2 \cdot 2! + (x+5)^3 \cdot 3! + \cdots$

is a power series having centre $C = -5$.

Convergence of a Power Series:

A power series is always convergent at its centre c.

Consider a power series $\sum_{n=0}^{\infty} a_n \cdot (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \text{---} \quad (1)$

Substituting $x = c$ in (1) we get:

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(c-c) + a_2(c-c)^2 + \text{---}$$
$$= a_0 + a_1(0) + a_2(0) + \text{---}$$
$$= a_0 \text{ (which is a finite sum \& hence convergent)}.$$

Radius of convergence (R.O.C) & Interval of convergence of a Power Series:

For a given power series $\sum_{n=0}^{\infty} a_n \cdot x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{---}$

exactly one of the following is true:

① The P.S. converges only at its centre 0 (ie $x = 0$)

or

② The P.S. converges absolutely $\forall x \in R$.

or

③ $\exists$ a real no. $R > 0$ such that the P.S. converges absolutely for $|x| < R$ & diverges for $|x| > R$. This R is called the radius of convergence of power series & the set of all possible values of x for which the given power series is convergent forms the Interval of convergence.

Hence, we can say that :

1. If the P.S converges only its centre $x=0$, then the $\boxed{R.O.C = 0 \ \& \ I.O.C = \{0\}}$

2. If the P.S converges $\forall x \in R$, then the $\boxed{R.O.C = \infty}$ & $\boxed{I.O.C = (-\infty, \infty)}$

3. If the P.S converges for some $(R \neq 0, \neq \infty)$ such that P.S is abs. convgt for $|x| < R$ & divgt for $|x| > R$

Then $\boxed{R.O.C = R}$ & $\boxed{I.O.C = (-R, R) \text{ or } [-R, R) \text{ or } (-R, R] \text{ or } [-R, R]}$
$(\neq 0, \neq \infty)$

(depending upon the fact that whether given P.S is convergent at end points or not).

Formula for finding the Radius of Convergence (R.O.C) :
(Derived from Ratio Test for Infinite Series).

Radius of convergence of a Power series (R.O.C) is given by:

$$\boxed{R = \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}} \ **$$

or

$$\boxed{R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|} \ **$$

Illustrations on Finding R.O.C of Power Series:

Q1. $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n. = \sum_{n=1}^{\infty} a_n \cdot x^n$ where $a_n = \frac{1}{n^2}$

$\therefore a_{n+1} = \frac{1}{(n+1)^2}$

R.O.C:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1/n^2}{1/(n+1)^2} \right| = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^2$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^2$$

$$= 1 \quad [\because \text{As } n\to\infty, \tfrac{1}{n}\to 0]$$

$\therefore$ R.O.C $= 1$.

$\Rightarrow$ Given P.S is convgt for $|x| < 1$ & divgt for $|x| > 1$.
(Refer Pg 2, (3))

Q2 $\sum_{n=1}^{\infty} \frac{n^n}{n!} \cdot x^n = \sum_{n=1}^{\infty} a_n \cdot x^n$ where $a_n = \frac{n^n}{n!}$

$\therefore a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

R.O.C:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^n / n!}{(n+1)^{n+1} / (n+1)!} \right| = \lim_{n\to\infty} \frac{n^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n \cdot \frac{\cancel{(n+1)} \cdot \cancel{n!}}{\cancel{n!} \cdot \cancel{(n+1)}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^n}}{\cancel{n^n}\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e}$$

$\therefore$ R.O.C $= \frac{1}{e}$

$\Rightarrow$ Given P.S is convgt for $|x| < \frac{1}{e}$ & divgt for $|x| > \frac{1}{e}$.

(Refer Pg 2, (3))

Q3: $\sum_{n=0}^{\infty} (-1)^n \cdot n! \cdot x^n = \sum_{n=0}^{\infty} a_n \cdot x^n$ where

$a_n = (-1)^n \cdot n!$

$\therefore a_{n+1} = (-1)^{n+1} \cdot (n+1)!$

R.O.C: $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

$$= \lim_{n\to\infty} \left|\frac{(-1)^n \cdot n!}{(-1)^{n+1}(n+1)!}\right|$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cdot \cancel{n!}}$$

$$= 0.$$

$\therefore$ R.O.C $= 0$

$\Rightarrow$ Given P.S is only convergent at its centre 0

& hence the I.O.C of given P.S $= \{0\}$ [Refer Pg 3, (1)]

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Q4. $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^3} \cdot x^n \pm \sum_{n=0}^{\infty} a_n \cdot x^n$ where.

$$a_n = \frac{(2n)!}{(n!)^3}$$

$$\therefore a_{n+1} = \frac{(2n+2)!}{((n+1)!)^3}$$

R.O.C: $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

$$= \lim_{n\to\infty} \left| \frac{(2n)!}{(n!)^3} \times \frac{((n+1)!)^3}{(2n+2)!} \right|$$

$$= \lim_{n\to\infty} \frac{\cancel{(2n)!}}{\cancel{(n!)^3}} \times \frac{(n+1)^3 \cdot \cancel{(n!)^3}}{(2n+2)(2n+1)\cdot \cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{(2n+2)(2n+1)} \quad (\text{Dividing N \& D by } n^3)$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^3}{\left(\frac{2}{n^2}+\frac{2}{n^3}\right)+\left(\frac{2}{n^2}+\frac{1}{n^3}\right)}$$

$$= \frac{1}{0} = \infty$$

$\therefore$ R.O.C $= \infty$

$\Rightarrow$ Given P.S is convergent $\forall x \in R$

$\Rightarrow$ The I.O.C of given P.S is $(-\infty, \infty)$ (Refer Pg 2, ②).

We now discuss Illustrations on finding. R.O.C ($R \neq 0, \neq \infty$) & its corresponding I.O.C:

Q5. Find the R.O.C & I.O.C of given Power Series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \cdot x^n = \sum_{n=0}^{\infty} a_n . x^n \quad \text{where} \quad a_n = \frac{(-1)^n}{2^n}$$

$\boxed{\text{R.O.C:}}$

$$\therefore a_{n+1} = \frac{(-1)^{n+1}}{2^{n+1}}$$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^n}{2^n} \times \frac{2^{n+1}}{(-1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} = \lim_{n\to\infty} \frac{\cancel{2^n}.2}{\cancel{2^n}} = 2.$$

$\therefore \boxed{R.O.C = 2} \Rightarrow$ Given P.S is convgt for $|x| < 2$ & divgt for $|x| > 2$. (Refer Pg 2, (3))

$\boxed{\text{I.O.C}}$: Note: Possible I.O.C's $= (-2,2)$ or $[-2,2)$ or $(-2,2]$ or $[-2,2]$
(Refer Pg 3, (3))

At $x = -2$: Given Power series becomes:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \cdot (-2)^n = \sum_{n=0}^{\infty} \frac{\cancel{2^n}}{\cancel{2^n}} = \sum_{n=0}^{\infty} 1$$

$$= 1 + 1 + 1 + \cdots$$

which is a geometric series with $|r| = 1$ & hence is divergent.

$\therefore$ Given P.S. is divergent at $x=-2$.

Now consider $x=2$:

At $x=2$, Given Power series becomes:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\cancel{2^n}} \cdot \cancel{2^x} = \sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+1-1+\ldots$$

Which is a geometric series with $|r| = |-1| = 1$ & hence it is divergent

$\therefore$ Given P.S is divergent at $x=2$

Hence, $\boxed{\text{the I.O.C. of given P.S} = (-2,2)}$ Ans.

Q6. Find the R.O.C & I.O.C of $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2}$ [GE 2019]. 5Mks.

$$= \sum_{n+\cancel{1}=\cancel{1}} \frac{x^{(n+\cancel{1})-\cancel{1}}}{(n+1)^2} = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}} \text{ ★★}$$

$\therefore$ Given P.S. can be expressed as $\sum_{n=0}^{\infty} a_n \cdot x^n$ where

$$a_n = \frac{1}{(n+1)^2} \quad \therefore a_{n+1} = \frac{1}{(n+2)^2}$$

R.O.C: $\quad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

$$= \lim_{n\to\infty} \left| \frac{1}{(n+1)^2} \times \frac{(n+2)^2}{1} \right| = \lim_{n\to\infty} \frac{(n+2)^2}{(n+1)^2}$$

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$$= \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^2}{\left(1+\frac{1}{n}\right)^2}$$ (Dividing N & D by n^2)

$$= 1$$

$\therefore$ $\boxed{R.O.C = 1}$ $\Rightarrow$ Given P.S is convgt for $|x| < 1$ & drvgt for $|x| > 1$. (Refer Pg 2, ③)

I.O.C:

Possible I.O.C's $= (-1,1)$ or $[-1,1)$ or $(-1,1]$ or $[-1,1]$.
(Refer Pg 3, ③)

At $x = -1$:

Given P.S becomes: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$ which is an Alternating series convergent by Leibnitz test as shown below:

Above series is:

$\sum_{n=0}^{\infty} (-1)^n \cdot a_n$ where $a_n = \frac{1}{(n+1)^2}$

① Clearly. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(n+1)^2} = 0$

② $\langle a_n \rangle$ is mon. $\downarrow$:

We know that $n+1 < n+2 \;\forall n \in \mathbb{N}$

$\Rightarrow (n+1)^2 < (n+2)^2 \;\forall n \in \mathbb{N}$

$\Rightarrow \frac{1}{(n+1)^2} > \frac{1}{(n+2)^2} \;\forall n \in \mathbb{N}$

$\Rightarrow a_n > a_{n+1} \;\forall n \in \mathbb{N}$

$\Rightarrow \langle a_n \rangle$ is mon. $\downarrow$.

① & ② $\Rightarrow$ Given series is convergent by Leibnitz test at $x = -1$

At $x=1$:

Given P.S becomes: $\sum_{n=0}^{\infty} \frac{(1)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$

$= \sum_{n=0}^{\infty} \frac{1}{n^2(1+\frac{1}{n})^2}$

$= \sum_{n=0}^{\infty} a_n$ where

$a_n = \frac{1}{n^2(1+\frac{1}{n})^2}$

Now, Take b_n as:

$b_n = \frac{1}{n^2}$

Then,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\cancel{n^2}(1+\frac{1}{n})^2}}{\frac{1}{\cancel{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2} = 1$$ (which is finite & non-zero)

$\therefore$ By Limit Comparison Test, Given series is convergent at $x=1$.

Hence, The I.O.C of given power series $= [-1, 1]$ Ans.

Q7 Find the I.O.C & R.O.C of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot x^n$. (11)

Given P.S $= \sum_{n=1}^{\infty} a_n \cdot x^n$ where $a_n = \frac{(-1)^n}{n}$.

$\therefore a_{n+1} = \frac{(-1)^{n+1}}{(n+1)}$

R.O.C: $R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$

$= \lim_{n \to \infty} \left| \frac{(-1)^n}{n} \times \frac{(n+1)}{(-1)^{n+1}} \right|$

$= \lim_{n \to \infty} \left(\frac{n+1}{n} \right) = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right) = 1$.

$\therefore \boxed{R.O.C = 1}$

$\Rightarrow$ Given P.S is convgt for $|x| < 1$ & divgt for $|x| > 1$.
(Refer Pg 2 (3)).

I.O.C: Possible I.O.C's $= (-1,1)$ or $[-1,1)$ or $(-1,1]$ or $[-1,1]$ (Refer Pg 3 (3))

At $x=1$: Given P.S becomes:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot (1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

which is an alternating series which is convergent by Leibnitz test as shown below:

Above series: $\sum_{n=1}^{\infty} (-1)^n \cdot a_n$ where $a_n = \frac{1}{n}$

(1) clearly $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

(2) $\langle a_n \rangle$ is mon. $\downarrow$:

We know that $n < (n+1) \; \forall \; n \in \mathbb{N}$.

$\Rightarrow \frac{1}{n} > \frac{1}{(n+1)} \; \forall \; n \in \mathbb{N}$.

$\Rightarrow a_n > a_{n+1} \; \forall \; n \in \mathbb{N}$.

$\Rightarrow \langle a_n \rangle$ is mon. $\downarrow$

$\therefore$ Given series is convergent at $x = 1$

At $x = -1$: Given P.S. becomes:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (-1)^n = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

which is a p-series with $p = 1$ & hence is divergent.

$\therefore$ Given series is divergent at $x = -1$

$\therefore$ The I.O.C of given power series $= (-1, 1]$

Note: In All Above Questions, we discussed problems on finding the R.O.C & I.O.C for power series centred at 0, we next discuss a question for finding R.O.C & I.O.C of a power series centred at $(c \neq 0)$.

Q8. Find the R.O.C & I.O.C of $\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n}}$

Note: Given P.S is centred at 2 *

Given P.S : $\sum_{n=1}^{\infty} a_n \cdot (x-2)^n$ where $a_n = \frac{1}{\sqrt{n}}$.

$\boxed{R.O.C}$ $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{1}{\sqrt{n}} \Big/ \frac{1}{\sqrt{n+1}} \right|$

$= \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n}}$

$\boxed{R = 1}$ $\therefore$ $\boxed{R.O.C = 1}$ Ans

$\therefore$ Given P.S. is convgt for $|x-2| < 1$

& divergent for $|x-2| > 1$ (Refer Pg 2, ③)

ie Given P.S is convergent for $-1 < x-2 < 1$

ie $-1+2 < x < 1+2$

ie $1 < x < 3$

Thus, Given P.S is convergent for $1 < x < 3$

$\boxed{I.O.C}$ Possible I.OC's = $(1,3)$ or $[1,3)$ or $(1,3]$ or $[1,3]$

(Refer Pg 3, ③)

$\boxed{\text{At } x=1:}$ Given P.S. becomes:

$$\sum_{n=1}^{\infty} \frac{(1-2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$ which is an alternating series which is convergent by Leibnitz test as shown below:

Above series: $\sum_{n=1}^{\infty} (-1)^n a_n$ where

$$a_n = \frac{1}{\sqrt{n}}.$$

(1) clearly $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}}$

$$= 0$$

(2) $\langle a_n \rangle$ is mon. $\downarrow$:

We know that $n < (n+1) \; \forall n \in \mathbb{N}$

$$\Rightarrow \sqrt{n} < \sqrt{n+1} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow a_n > a_{n+1} \quad \forall n \in \mathbb{N}.$$

$\therefore$ Given series is convgt at $x=1$ by Leibnitz test.

$\boxed{\text{At } x=3:}$ Given P.S. becomes:

$$\sum_{n=1}^{\infty} \frac{(3-2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$ which is a p-series having $p = \frac{1}{2} < 1$ & hence is divergent

$\therefore$ Given series is divergent at $x=3$.

$\therefore$ I.O.C of Power Series is: $[1, 3)$ Ans

Practice Problems :

Find the R.O.C & I.O.C. of following Power series :

Answers :

1. $\sum_{n=1}^{\infty} (-1)^n \cdot n \cdot x^n$ — R.O.C $=1$, I.O.C $=(-1,1)$
2. $\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$ — R.O.C $=\infty$, I.O.C $=(-\infty,\infty)$
3. $\sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot x^n$ — R.O.C $=\infty$, I.O.C $=(-\infty,\infty)$
4. $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n!}$ — R.O.C $=\infty$, I.O.C $=(-\infty,\infty)$
5. $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (x-1)^n}{n}$ — R.O.C $=1$, I.O.C $=(0,2]$
6. $\sum_{n=1}^{\infty} \frac{x^n}{3^n \cdot n}$ — R.O.C $=3$, I.O.C $=[-3,3]$
7. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$ — R.O.C $=1$, I.O.C $=[4,6]$
8. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ — R.O.C $=\infty$, I.O.C $=(-\infty,\infty)$
9. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n} \cdot x^n$ — R.O.C $=1$, I.O.C $=(-1,1]$
10. $\sum_{n=0}^{\infty} \frac{x^n}{2n+3}$ — R.O.C $=1$, I.O.C $=[-1,1)$

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