Department of Mathematics

GE-4 Elements of Analysis, Semester IV (CBCS)

Unit-1

Sets and functions

Assignment

Q1) Determine which of the following sets are finite, countably infinite or countable. Give reasons.

(a) $\{ \frac{1}{n} : n \in \mathbb{Z} \setminus \{0\} \}$

(b) $\{ x \in \mathbb{N} : |x - 20| > |x| \}$

(c) The set of all polynomials with integer coeffecients.

(d) $\{1, 4, 9, 16, \ldots\}$

(e) $\{ m \in \mathbb{Z} \mid m \equiv 2 \pmod 3 \}$

(f) $\{ n \in \mathbb{Z} : n \geq 5 \}$

Q2) State whether the following are true or false. Give reasons

(a) If a set A is denumerable, then it is always infinite.

(b) If a set A is countable, then it is always infinite.

(c) Every subset of a denumerable set is denumerable.

(d) Every subset of a countable set is countable.

(e) Every superset of a finite set is finite.

(f) Union of every countable collection of finite sets is finite.

The Algebraic and Order Properties of Real Numbers

Assignment

Attempt questions (1)-(4) using the Order Properties of Real Numbers

Q1) If $0 < a < b$, show that $a < \sqrt{ab} < b$

Q2) Let $a > 0$ and $b > 0$, then show that $\sqrt{ab} \leq \frac{1}{2}(a+b)$ with equality occurring if and only if $a = b$

Q3) Let $S = \{ a + b\sqrt{5} : a, b \in \mathbb{Q} \}$. Show that S satisfies the following:

(a) If $x_1, x_2 \in S$, then $x_1 + x_2 \in S$ and $x_1 x_2 \in S$

(b) If $x \neq 0$ and $x \in S$, then $1/x \in S$

Q4) Find all real numbers x which satisfy the following inequalities:

(a) $2x+3 \leq 6$ (b) $x^2 + x > 2$ (c) $\frac{2x+1}{x+2} < 1$

The Order Properties of $\mathbb{R}$

There is a non-empty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties:

(i) If $a, b \in \mathbb{P}$, then $a+b \in P$

(ii) If $a, b \in \mathbb{P}$, then $a \cdot b \in P$

(iii) If $a \in \mathbb{R}$, then exactly one of the following holds:

$$a \in \mathbb{P}, \quad a = 0, \quad -a \in \mathbb{P}$$

The property (iii) is called the Trichotomy property. From this it follows that the set of real numbers can be written as a union of three disjoint sets

$$\text{ie} \quad \mathbb{R} = \mathbb{P} \cup \{0\} \cup \{-a : a \in \mathbb{P}\}$$

This further classifies $\mathbb{R}$ into numbers that are positive (ie when $a \in \mathbb{P}$), non-negative (when $a \in P \cup \{0\}$), negative (when $-a \in \mathbb{P}$) and non-positive (when $-a \in \mathbb{P} \cup \{0\}$)

Absolute Value

The absolute value of a real number a, denoted by $|a|$, is defined by

$$|a| = \begin{cases} a, & \text{if } a > 0 \\ 0, & \text{if } a = 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Theorem

(a) $|ab| = |a|\,|b| \quad \forall\, a, b \in \mathbb{R}$ (b) $|a|^2 = a^2 \quad \forall\, a \in \mathbb{R}$

(c) If $c \geq 0$, then $|a| \leq c$ iff $-c \leq a \leq c$

(d) $-|a| \leq a \leq |a| \quad \forall\, a \in \mathbb{R}$

Triangular Inequality

$$|a+b| \leq |a| + |b| \quad \forall\, a, b \in \mathbb{R}$$

Corollary

(a) $||a| - |b|| \leq |a-b| \quad \forall\, a, b \in \mathbb{R}$ (b) $|a-b| \leq |a| + |b| \quad \forall\, a, b \in \mathbb{R}$

(c) $|a_1 + a_2 + a_3 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| \quad \forall\, a_1, a_2, \ldots a_n \in \mathbb{R}$

Find all $x \in \mathbb{R}$ that satisfy the following inequalities:

Q1) $|4x - 5| \leq 13$

Solution:

Using theorem (c), we have $|4x-5| \leq 13$

$$\iff -13 \leq 4x - 5 \leq 13$$
$$\iff -8 \leq 4x \leq 18$$
$$\iff -2 \leq x \leq 9/2 \quad [\text{Dividing by } 4]$$

Hence $x \in A = \{x \in \mathbb{R} : -2 \leq x \leq 9/2\}$

Q2) $4 < |x+2| + |x-1| < 5$

Solution: Consider the following cases:

(i) $x < -2$

Here $|x+2| = -(x+2)$ and $|x-1| = -(x-1)$

$\therefore$ The inequality reduces to

$$4 < -(x+2) - (x-1) < 5$$
$$\iff 4 < -2x - 1 < 5 \iff 5 < -2x < 6$$

Hence $\quad -3 < x < -5/2$

(ii) $\quad -2 \le x < 1$

Here $|x+2| = x+2$ and $|x-1| = -(x-1)$

Thus the inequality becomes

$$4 < (x+2) - (x-1) < 5$$

$$\Leftrightarrow 4 < 3 < 5$$

which is not true

Hence there is no $x \in \mathbb{R}$ such that $-2 \le x < 1$ and x satisfies the inequality.

(iii) $\quad x \ge 1$

Here $|x+2| = x+2$ and $|x-1| = x-1$

Thus the inequality becomes

$$4 < (x+2) + (x-1) < 5$$

$$\Leftrightarrow 4 < 2x+1 < 5$$

$$\Leftrightarrow 3 < 2x < 4$$

$$\Leftrightarrow 3/2 < x < 2$$

Combining (i), (ii) and (iii), the values of x which satisfy the inequality belong to the set $A = \{x \in \mathbb{R} : -3 < x < -5/2$ or $3/2 < x < 2\}$

Q3) $|2x-3| < 5$ and $|x+1| > 2$ simultaneously

Q4) $|x+1| + |x-2| = 7$

Q5) Show that if $a, b \in \mathbb{R}$ then

(a) $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$

(b) $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$

Suprema and Infima

Definitions:

Let S be a non-empty subset of $\mathbb{R}$

1) Bounded above set

The set S is said to be a bounded above set if $\exists\ u \in \mathbb{R}$ such that $x \leq u \quad \forall\ x \in S$.

The number u is called an upper bound of S

2) Bounded below set

The set S is said to be a bounded below set if $\exists\ l \in \mathbb{R}$ such that $l \leq x \quad \forall\ x \in S$.

The number l is called a lower bound of S

3) Bounded Set

The Set S is said to be bounded if it is both bounded above and bounded below.

A set that is not bounded is said to be an unbounded set.

4) Supremum

If S is bounded above, then a number u is said to be a supremum (or a least upper bound) of S if it satisfies the following conditions:

(i) u is an upper bound of S, and

(ii) If v is any upper bound of S, then $u \leq v$

5) Infimum

If S is bounded below, then a number l is said to be an infimum (or a greatest lower bound) of S if it satisfies the following conditions:

(i) l is a lower bound of S, and

(ii) If w is any lower bound of S, then $w \leq l$

Note: If the supremum and Infimum of a set S exist, they are unique and are denoted by $\sup S$ and $\inf S$

The Completeness Property of $\mathbb{R}$

Every nonempty subset S of $\mathbb{R}$ which is bound above has a supremum in $\mathbb{R}$.
The analogous property for infima states that every nonempty subset S of $\mathbb{R}$ that is bounded below has an infimum in $\mathbb{R}$.

The Archimedean Property

If $x \in \mathbb{R}$, then $\exists$ $n_x \in \mathbb{N}$ such that $x < n_x$

Find inf S and sup S for the following:

Q1) $S = \left\{ \frac{4n+3}{n} : n \in \mathbb{N} \right\}$

Solution

Clearly the set S is non-empty.
The set S can be written as

$$S = \left\{ 4 + \frac{3}{n} : n \in \mathbb{N} \right\} = \left\{ 7, \frac{11}{2}, 5, \frac{19}{4}, \dots \right\}$$

(a) Upper bound and Supremum.

Obviously, $4 + \frac{3}{n} \leq 7 \quad \forall\, n \in \mathbb{N}$

Hence 7 is an upper bound of S
By Order Completeness property of $\mathbb{R}$, S has a Supremum in $\mathbb{R}$.

Claim: $\sup S = 7$

Since 7 is an upper bound of S, hence by definition
$\sup S \leq 7$

Also $\because 7 \in S \quad \therefore 7 \leq \sup S$

Hence $\sup S = 7$

(b) Lower bound and Infimum

Obviously $4 \leq 4 + \frac{3}{n} \quad \forall\, n \in \mathbb{N}$

Hence 4 is a lower bound of S

By the Order Completeness property of $\mathbb{R}$, S has an infimum in $\mathbb{R}$.

Claim : $\text{Inf } S = 4$

Let us assume to the contrary, i.e., 4 is not the greatest lower bound.

Hence $\exists\ v \in \mathbb{R}$ such that v is a lower bound of S and

$$v > 4$$

$$\Rightarrow \quad v - 4 > 0$$

Hence by the Archimedean Property for $x = \frac{3}{v-4} \in \mathbb{R}$,

$\exists\ n_x \in \mathbb{N}$ such that

$$n_x > \frac{3}{v-4}$$

$$\Rightarrow \quad v > 4 + \frac{3}{n_x}$$

This contradicts the fact that v is a lower bound of S

Hence $\text{Inf } S = 4$

Q2) $S = \left\{ -2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, \dots, -\frac{(n+1)}{n}, \dots \right\}$

Q3) $S = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$