B.A. (Hons.) Economics - Semester IV

Supplementary Study Material on Unit III: Taylor & MacLaurin Series

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GE-IV : Elements of Analysis

B.A (Hons) Economics - Semester IV

Unit III : Taylor and MacLaurin Series Expansion :

If a function is represented by power series:

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots}$$

having radius of convergence $R>0$, centred at c; then its coefficients are given by $\boxed{a_n = \frac{f^n(c)}{n!}}$

Hence, above function can be expressed as: ✶✶

$$\boxed{f(x) = f(c) + f'(c).(x-c) + \frac{f''(c)}{2!}.(x-c)^2 + \frac{f'''(c)}{3!}.(x-c)^3 + \frac{f^n(c)}{n!}.(x-c)^n + \cdots}$$

└(I)

Equation (I) represents the Taylor series expansion for a function $f(x)$ centred at c.

Note: In Particular if $c=0$; the function is centred at 0, then substituting $c=0$ in (I), we get: ✶✶

$$\boxed{f(x) = f(0) + f'(0).x + \frac{f''(0)}{2!}.x^2 + \frac{f'''(0)}{3!}.x^3 + \cdots \frac{f^n(0)}{n!}.x^n + \cdots}$$

Equation (II) represents the MacLaurin Series expansion for a function $f(x)$ centred at 0. └(II)

Notes:

Students are advised to memorise equations (I) & (II) as formulas to be used directly for solving problems on Taylor & MacLaurin Series expansions.

Illustrations on Taylor Series expansion:

Q1. Find the Taylor series expansion for $f(x) = \frac{1}{x}$ at $x = 2$. Also discuss the convergence of the series.

Sol: $f(x) = \frac{1}{x}$, $c = 2$. $\Rightarrow f(2) = \frac{1}{2}$

$f'(x) = -\frac{1}{x^2} \Rightarrow f'(2) = -\frac{1}{4}$

$f''(x) = \frac{2}{x^3} \Rightarrow f''(2) = \frac{2}{8} = \frac{1}{4}$

$f'''(x) = \frac{-2 \cdot 3x^2}{x^6} = \frac{-6}{x^4} \Rightarrow f'''(2) = \frac{-6}{16} = \frac{-3}{8}$

⋮

& so on...

Taylor series expansion of $f(x)$ is given by:

$$\boxed{f(x) = f(2) + f'(2)\cdot(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots}$$

$$\Rightarrow \frac{1}{x} = \frac{1}{2} + \left(-\frac{1}{4}\right)(x-2) + \frac{1}{4}\times\frac{1}{2}(x-2)^2 + \left(-\frac{\cancel{3}}{8}\right)\cdot\frac{1}{\cancel{6}\,2}(x-2)^3 + \cdots$$

$$\Rightarrow \frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{4} + \frac{(x-2)^2}{8} - \frac{1}{16}(x-2)^3 + \cdots$$

$$\Rightarrow \frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \frac{(x-2)^3}{2^4} + \cdots$$

$$\Rightarrow \frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x-2)^n}{2^{n+1}}$$

We also observe that the above series is a geometric series with first term $a = \frac{1}{2}$ & common ratio $r = \frac{-(x-2)}{2}$

& is hence convergent if $|r| = \left|-\frac{(x-2)}{2}\right| < 1$

$$\Rightarrow |x-2| < 2$$

$$\Rightarrow 0 < x < 4$$

The sum of above series is given by $\frac{a}{1-r}$

$$= \frac{\frac{1}{2}}{1 + \frac{x-2}{2}} = \frac{1}{x}.$$

Hence, It is Verified that the above series is convergent to $\frac{1}{x}$ for $0 < x < 4$.

Q2 Using the Taylor's Expansion, expand $\cos x$ in powers of $\left(x-\frac{\pi}{2}\right)$.

Sol: here $f(x) = \cos x$. & $c = \frac{\pi}{2}$.

$f(x) = \cos x \Rightarrow f(c) = f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

$f'(x) = -\sin x \Rightarrow f'(c) = f'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

$f''(x) = -\cos x \Rightarrow f''(c) = f''\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0$

$f'''(x) = \sin x \Rightarrow f'''(c) = f'''\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$.

$f^{iv}(x) = \cos x \Rightarrow f^{iv}(c) = f^{iv}\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

& so on...

Taylors' Expansion is given by:

$$\boxed{f(x) = f(c) + (x-c)\cdot f'(c) + \frac{(x-c)^2}{2!}\cdot f''(c) + \frac{(x-c)^3}{3!}\cdot f'''(c) + \text{---}}$$

$$\Rightarrow \cos x = 0 + \left(x-\frac{\pi}{2}\right)\cdot(-1) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}\cdot 0 + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!}\cdot(1) + \text{---}$$

$$\Rightarrow \cos x = -\left(x-\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!} + \text{---}$$ Ans.

Q3 Find Taylor's expansion of e^x in powers of $(x-2)$. ⑤

Sol: $f(x) = e^x$, $c = 2$.

$f(2) = e^2$

$f'(x) = e^x \Rightarrow f'(2) = e^2$

$f''(x) = e^x \Rightarrow f''(2) = e^2$

$f'''(x) = e^x \Rightarrow f'''(2) = e^2$

& so on.

∴ Taylor's Expansion is given by:

$$f(x) = f(2) + (x-2) \cdot f'(2) + \frac{(x-2)^2}{2!} \cdot f''(2) + \frac{(x-2)^3}{3!} \cdot f'''(2) + \cdots$$

$$\Rightarrow e^x = e^2 + (x-2) \cdot e^2 + \frac{(x-2)^2}{2!} e^2 + \frac{(x-2)}{3!} \cdot e^2 + \cdots$$

$$\Rightarrow e^x = e^2 \left[1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \cdots \right]$$

Ans.

Q4) Expand $\sin x$ in powers of $(x - \frac{\pi}{2})$ & prove:

$$\sin x = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} + \cdots$$

(Do Yourself!!).

Illustrations on Mac Laurin's Expansion: ** [V. Imp].

Q.1 Find Mac' Laurins' Expansion of $\cos x$.

Sol: $f(x) = \cos x$, $c = 0$

$\Rightarrow f(0) = \cos 0 = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = \sin 0 = 0$

$f^{IV}(x) = \cos x \Rightarrow f^{IV}(0) = \cos 0 = 1$

$f^{V}(x) = -\sin x \Rightarrow f^{V}(0) = -\sin 0 = 0.$

$f^{VI}(x) = -\cos x \Rightarrow f^{VI}(0) = -\cos 0 = -1$

Mac Laurins' Expansion is given by:

$$\boxed{f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} \cdot f'''(0) + \frac{x^4}{4!} \cdot f^{IV}(0) + \cdots}$$

$$\Rightarrow \cos x = 1 + x \cdot 0 + \frac{x^2}{2!} \cdot (-1) + \frac{x^3}{3!} \cdot (0) + \frac{x^4}{4!} \cdot (1) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-1) + \cdots$$

$$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\Rightarrow \cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} \quad \text{Ans}$$

Q2 Find Mac Laurins' Expansion of $\sin x$.

[GE-2017 5Mks]

Sol: $f(x) = \sin x \; ; \; c = 0$

$\Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$

$f^{IV}(x) = \sin x \Rightarrow f^{IV}(0) = \sin 0 = 0$

$f^{V}(x) = \cos x \Rightarrow f^{V}(0) = \cos 0 = 1$

$f^{VI}(x) = -\sin x \Rightarrow f^{VI}(0) = -\sin 0 = 0$

$f^{VII}(x) = -\cos x \Rightarrow f^{VII}(0) = -\cos 0 = -1$

Mac Laurins' Expansion is given by:

$$\boxed{f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} \cdot f''(0) + \frac{x^2}{3!} \cdot f'''(0) + \frac{x^4}{4!} \cdot f^{IV}(0) + \frac{x^5}{5} \cdot f^{V}(0) + \cdots}$$

$$\Rightarrow \sin x = 0 + x \cdot 1 + \frac{x^2}{2!}(0) + \frac{x^3}{3!} \cdot (-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \frac{x^6}{6!}(0) + \frac{x^7}{7!} \cdot (-1) + \cdots$$

$$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\Rightarrow \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{Ans}$$

Q3 Obtain Mac Laurins' Expansion of e^x. (GE-2018, 2019 5 Marks)

Sol:- $f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1$

$f^{iv}(x) = e^x \Rightarrow f^{iv}(0) = e^0 = 1$

& so on.....

Mac Laurins' Expansion is given by:

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} \cdot f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \text{---}$$

$$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \text{---}$$

$$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ Ans.

Q4. Obtain the Mac Laurins expansion of $\log(1+x)$.

Sol; $f(x) = \log(1+x) \Rightarrow f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = \frac{1}{1+0} = 1$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$

$f^{iv}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{iv}(0) = -6$

$$f^{v}(x) = \frac{24}{(1+x)^5} \Rightarrow f^{v}(0) = 24$$

& so on . . . . .

Mac Laurins expansion is given by:

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} \cdot f''(0) + \frac{x^3}{3!} \cdot f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} \cdot f^{v}(0) + - - -$$

$$\Rightarrow \log(1+x) = 0 + x \cdot (1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \frac{x^5}{5!}(24) + - - -$$

$$\Rightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - - - -$$

$$\Rightarrow \log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n} . \text{ Ans}$$

Q5 Obtain Mac Laurins Expansion of $\tan^{-1}x$. (10)

Sol: $f(x) = \tan^{-1}x \Rightarrow f(0) = \tan^{-1}(0)$
$= 0$

$$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(x) = \frac{1}{1+0} = 1.$$

$$f''(x) = \frac{-2x}{(1+x^2)^2} \Rightarrow f''(0) = \frac{-2(0)}{[1+0]^2} = \frac{0}{1} = 0$$

$$f'''(x) = -2\left[\frac{(1+x^2)^2 - x \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4}\right]$$

$$= \frac{-2\left[(1+x^2)^2 - 4x^2(1+x^2)\right]}{(1+x^2)^4}$$

$$= \frac{-2\,\cancel{(1+x^2)}\left[1+x^2 - 4x^2\right]}{(1+x^2)^{\cancel{4}}}$$

$$= \frac{-2(1-3x^2)}{(1+x^2)^3}$$

$$\therefore f'''(x) = \frac{-2(1-3x^2)}{(1+x^2)^3}$$

$$\Rightarrow f'''(0) = \frac{-2(1-0)}{(1+0)^3} = \frac{-2}{1} = -2$$

$$f^{iv}(x) = -2\left[\frac{(1+x^2)^3 \cdot (-6x) - (1-3x^2) \cdot 3(1+x^2) \cdot 2x}{(1+x^2)^6}\right]$$

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$$f^{iv}(x) = -2 \cdot \left[\frac{-6x(1+x^2)^3 - 6x(1+x^2)(1-3x^2)}{(1+x^2)^6} \right]$$

$$\Rightarrow f^{iv}(0) = -2 \left[\frac{0-0}{1} \right] = 0$$

$\therefore$ Mac Laurins' Expansion is given by:

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + -$$

$$\Rightarrow \tan^{-1} x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot (-2) + \frac{x^4}{4!}(0) + - -$$

$$\Rightarrow \tan^{-1} x = x - \cancel{2} \cdot \frac{x^3}{3 \times \cancel{2} \times 1} + \cdot$$

$$\Rightarrow \tan^{-1} x = x - \frac{x^3}{3} + - - .$$

Practise Problems:

Q. Obtain the MacLaurins expansion of following & verify:

1) $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots$

2) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

3) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$

4) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

5) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$