MATHEMATICS OF FINANCE – PART 3

ASHA RANI

B.COM(HONS.) IVth SEMESTER

BUSINESS MATHEMATICS (PAPER- BCH 4.2), SECTION A

Session Details

This session covers the following:

- Annuity Due
- Deferred annuity
- Perpetuity
- Application of ordinary annuity for Sinking Fund

Annuity Due

Annuity due is an annuity whose payment is due immediately at the beginning of each interval.

Each annuity payment will compound for one extra period. Thus, the present and future values of an annuity-due will be calculated accordingly.

- Examples of annuity due are –
  Deposits in savings,
  Rent or lease payments,
  Insurance premiums.

Future value of Annuity Due

The formula for the **future value** of an **Annuity Due**

$$FV_n = R\frac{(1+i)^{n+1}-1}{i} - 1$$

- OR

$$FV_n = R\ (s_{n+1}\neg i\ \text{-}1)$$

- Where,
- FV_n =Future value of annuity at the end of each period
- R= Regular payment at the end of each payment interval
- i = interest rate per period
- n = number of period for which annuity will lasts
- $s_n\neg i$ = tabulated value of future value of Interest rate factor annuity (FVIFA)

Present value of Annuity Due

The formula for the Present **value** of an **Annuity Due**

$$PV_n = R\frac{[1-(1+i)^{-n}]}{i} * (1+i)$$

- OR

$$PV_n = R\,(a_{n-1}\neg i + 1)$$

- Where,
- PV_n =Present value of annuity at the end of each period
- R= Regular payment at the end of each payment interval
- i = interest rate per period
- n = number of period for which annuity will lasts
- $a_n\neg i$ = tabulated value of present value of Interest rate factor annuity (PVIFA)

DERIVATION OF FORMULAE FOR ANNUITY DUE

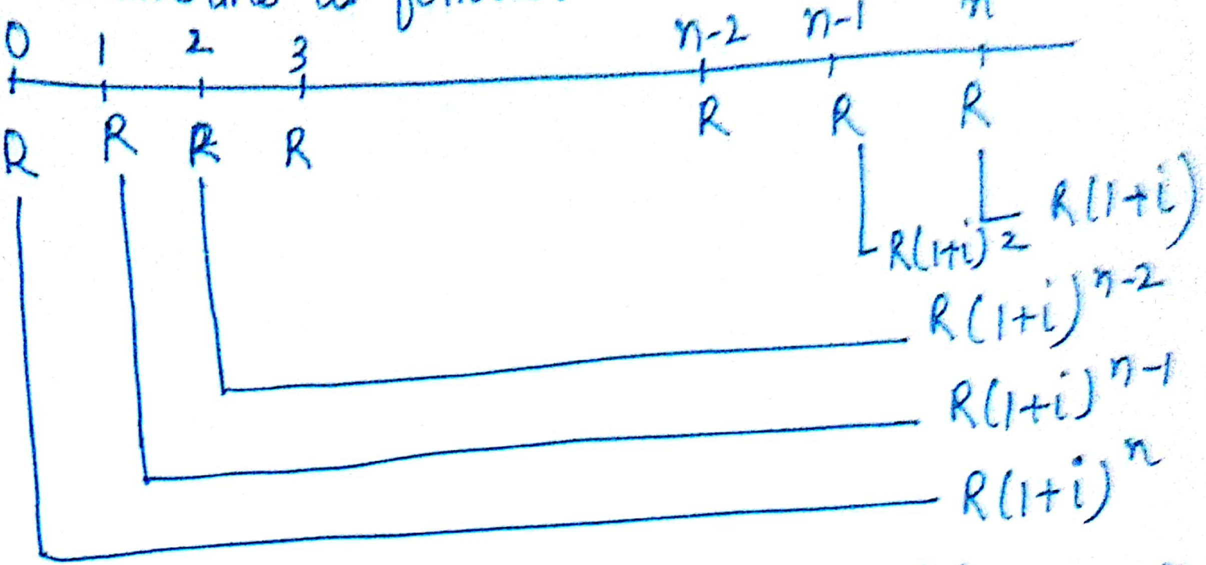
Future value

Present value

Derivation of future Value of Annuity Due

Let ₹ R be the payment made at the beg of each year for n years at i % interest rate compounded annually.

Then the series of payment can be shown is the time line as follows:



$$F.V = R(1+i) + R(1+i)^2 + \cdots + R(1+i)^{n-1} + R(1+i)^n$$

$$= R(1+i)\left[1 + (1+i)^1 + \cdots + (1+i)^{n-1}\right]$$ → This is geometric series the we can write → $\left[\frac{r^n - 1}{r-1}\right]$

$$= R(1+i)\left[\frac{(1+i)^n - 1}{\cancel{1}+i-\cancel{1}}\right]$$

$$= R\left[\frac{(1+i)^n \cdot (1+i) - (1+i)}{i}\right] = R\left[\frac{(1+i)^{n+1} - 1}{i} - \frac{i}{i}\right]$$

$$= R\left[\frac{(1+i)^{n+1} - 1}{i} - 1\right] \quad \text{OR} \quad R\left[s_{\overline{n+1}|i} - 1\right]$$

Similar way we can derive P.V formula for Annuity Due.

ILLUSTRATIVE PROBLEMS

Mr. X purchases a house for Rs.3,00,000. He agrees to pay for the house in 5 equal installment in the beginning of each year. If money is worth 5% effective, what should be the size of each installment.

Mr. X wants Rs.5,00,000 at the end of 7 years. If the rate of interest is 8%, what amount shall be deposited at the beginning of each quarter to get this amount.

Solution :- Let the size of each installment be ₹R

P.V. of the house is ₹ 3,00,000 & payment is made at the beg.

$\therefore PV = R(a_{\overline{n-1}|i} + 1)$ $\qquad n = 5$, $r = .05$

$3,00,000 = R(a_{\overline{4}|.05} + 1)$

$R = \dfrac{3,00,000}{3.54595050 + 1} = ₹ 43,995.20$

$\therefore$ size of installment is ₹ 43,995.20

<u>Solution</u>:- Let the amount deposited at the beg. of each quarter be ₹R

F.V. after 7 years is ₹5,00,000

$n = 7 \times 4 = 28$

$FV = R\left(S_{\overline{n+1}|i} - 1\right)$

$r = \frac{.08}{4} = .02$

$5,00,000 = R\left(S_{\overline{28+1}|.02} - 1\right)$

$R = \frac{5,00,000}{38.79223451 - 1} = ₹13230.23$

∴ the amount of Quaterly deposit is

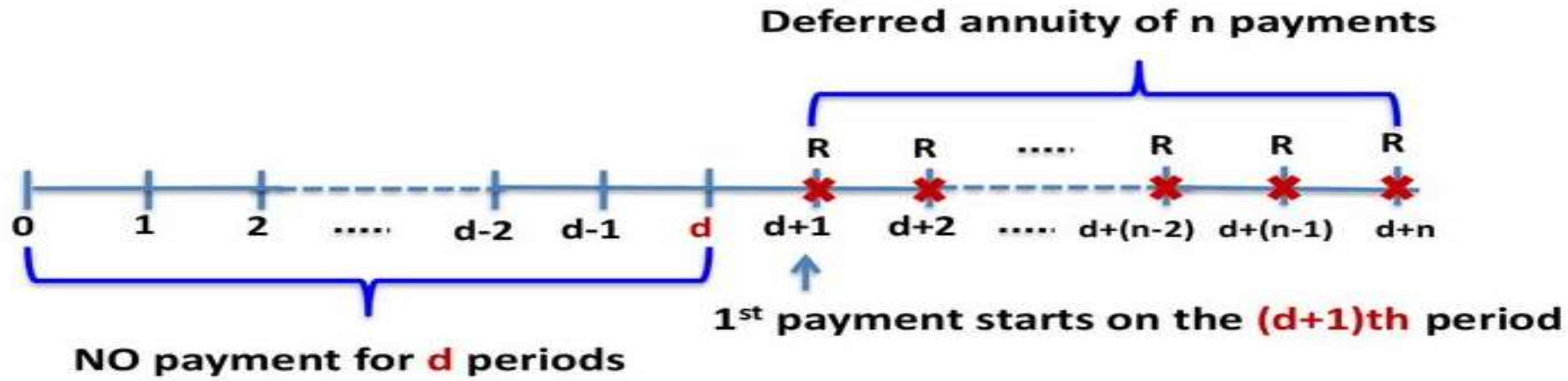
₹ 13,230.23

DEFERRED ANNUITY

A **deferred annuity** is that type of annuity where payments are delayed until the expiration of certain time period. This implies that the term of annuity will continue and become operational after d years(where d demotes deferred period)

Usually the annuity has two stages, as depicted in this figure in the next slide first is deferred period and the second is annuity payment period.

DEFERRED ANNUITY:



Future value of Deferred Annuity

The formula for the future value of a Deferred Annuity is same as the formula for F.V. of ordinary annuity . As the situation is same because if the annuity start at 0 period or after d period, we are anyways interested to find the future value of total n payment.

$$FV_n = R\ (s_n \neg i)$$

- Where,
- FV_n =Future value of deferred annuity
- R= Regular payment at the end of each payment interval
- i = interest rate per period
- n = number of period for which annuity will lasts
- $s_n \neg i$ = tabulated value of future value of Interest rate factor annuity (FVIFA)

Present value of Deferred Annuity

The formula for the Present **value** of **Deferred Annuity is computed as a difference between the present value computed for entire period (deferred period plus annuity payment period) and present value computed for deferred period (where no payment is made)**

$$PV_n = R\,(a_{d+n}\neg i - a_{d\neg i})$$

- Where,
- PV_n =Present value of deferred annuity
- R= Regular payment at the end of each payment interval
- i = interest rate per period
- n = number of period for which annuity will lasts
- $a_{d+n}\neg i$ = present value of a rupee 1 for d+n period (entire period)
- $a_{d\neg i}$ = present value of a rupee 1 for deferred period

ILLUSTRATIVE PROBLEM

A house is sold for Rs. 50,000 down and 10 semi-annual payments of Rs. 5,000 each, the first due 3 year hence. Find the cash price of the house if money is worth 20% compounded semi annually.

Mr. X borrows from bank Rs. 12,00,000 at the rate of 9% p.a. to be repaid in six equal instalments with interest the first instalment falling due at the end of 4^{th} year. Find the amount of each instalment.

Solution:- Let r be the rate of interest, d be the number of non-payment period / deferred period, n be the number of payment period.

Cash Price of the House = Down Payment + P.V. of annuity during the period between d & m.

$R = ₹ 5000 \quad i = \frac{.20}{2} = .10 \quad d = 2 \times 2 + 1 = 5 \quad n = 10$

Cash price of the house

$$= 50,000 + 5000 \left(a_{\overline{d+m}|i} - a_{\overline{d}|i} \right)$$

$$= 50,000 + 5000 \left(a_{\overline{15}|.10} - a_{\overline{5}|.01} \right)$$

$$= 50,000 + 5,000 \left(7.60607951 - 3.79078677 \right)$$

$$= 50,000 + 19,076.46$$

Cash Price of the House $= ₹ 69,076.46$

Solution :-

Let the amount of each installment be ₹ R

P.V = ₹ 12,00,000

$i = .09$ $d = 4-1 = 3$ $n = 6$

$$P.V = R\left(a_{\overline{d+n|}i} - a_{\overline{d|}i}\right)$$

$$12,00,000 = R\left(a_{\overline{3+6|}.09} - a_{\overline{3|}.09}\right)$$

$$R = \frac{12,00,000}{(5.99524689 - 2.53129467)}$$

$$= \frac{12,00,000}{3.4639522}$$

$$\boxed{R = ₹\,346425.1}$$

PERPETUAL ANNUITY

A *perpetuity* is an annuity for which the payments will continue forever.

In case of perpetuity, we cannot compute the amount because as time increases it goes beyond all bounds.

- Present value of perpetuity can be computed in these two cases:

A) PV where payment is made at the end

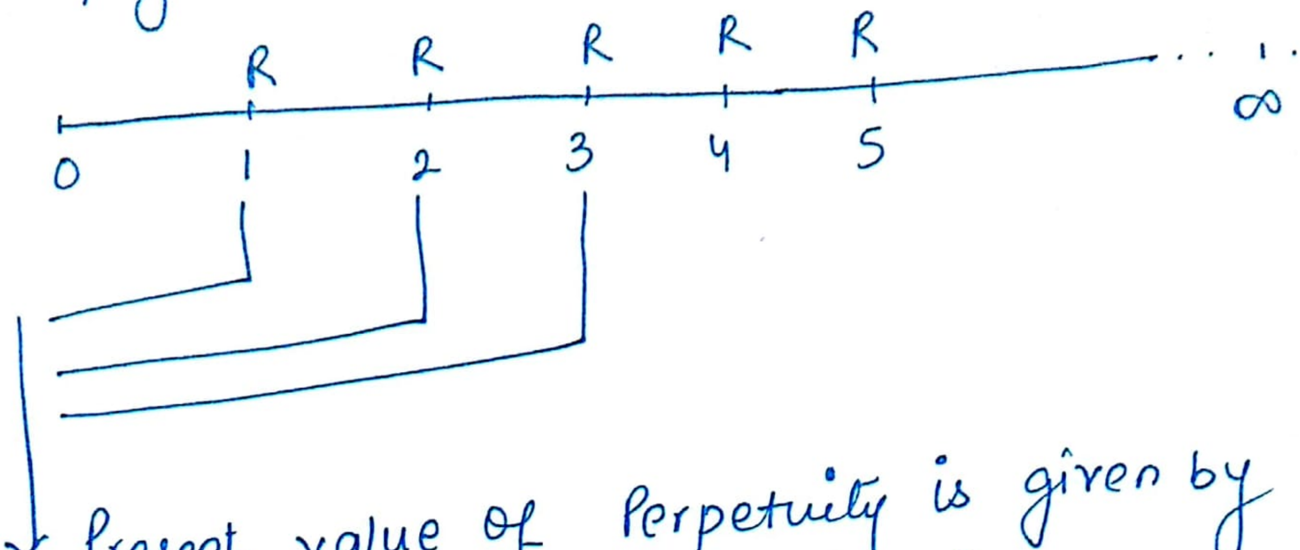
- $PV = \frac{R}{r}$

A) PV where payment is made at the Beginning

- $PV = R + \frac{R}{r}$

Derivation of P.V. of Perpetuity when Payment is made at the end.

Let R be the amount of Regular payment. & i be the interest rate.



Present value of Perpetuity is given by

$$P.V = R(1+i)^{-1} + R(1+i)^{-2} + R(1+i)^{-3} + \cdots \quad (1)$$

If we multiply eq-(1) by $(1+i)$ we get

$$P.V(1+i) = R + R(1+i)^{-1} + R(1+i)^{-2} + R(1+i)^{-3} + \cdots \quad (2)$$

Now if we subtract (1) eq. from (2)

$$P.V(1+i) - P.V = R$$

$$P.V[(1+i)-1] = R$$

$$P.V = \frac{R}{\cancel{1}+i-\cancel{1}} = \frac{R}{i}$$

If payment is made at the beg. then

$$P.V = R + \frac{R}{i}$$ (in similar way we can derive this)

ILLUSTRATIVE PROBLEM

How much is needed to endure a monthly pension of ₹60,000 at the beginning of each month indefinitely, if the money is worth 9% compounded monthly?

Sol^n:- let P be the money needed to ensure a monthly pension of ₹ 60000

$P = R + \frac{R}{r}$ $\Rightarrow$ $r = \frac{.09}{12} = .0075$

$R = 60000$

$P = 60000 + \frac{60000}{.0075} = 60000 + 8000000$

$\boxed{P = ₹8060000}$

Sinking Fund

Sinking fund refers to a fund created to set aside a fixed amount over time to accumulate a lumpsum amount at the end of specific period of time .

Sinking fund is also referred to as a fund created by a company to accumulate money for replacement of a large asset or any other major expenditure

SINKING FUND…

Suppose that the account has an annual interest rate of r compounded m times per year, so that $i = \frac{r}{m}$ is the interest rate per compounding period. If you make a payment of Rs. R at the end of each period or may be at the beginning , then the future value after years, or periods, will be

- $FV_n = R\,[\,((1+i)^n - 1)]/\,i$ OR $FV_n = R\,\frac{(1+i)^{n+1}-1}{i} - 1$

If we are interested to find out regular payment then

- R = $FV_n * \; i/\{(1+i)^n - 1\}$ or R= $FV_n[\frac{1}{\frac{(1+i)^{n+1}-1}{i}-1}]$

ILLUSTRATIVE PROBLEM

Mr. A plans to send his son to U. K. for higher education 10 year hence. He anticipates the cost at that time will be Rs.10,00,000.How much should he save at the beginning of each year to accumulate this amount at the end of 10 years if the rate of interest is 12% p.a. effective?

Solution:- Let ₹R be the annual amount to be saved & invested at the beg. of each year.

By applying future value formula

$F.V = R\, S_{\overline{n+1}|i}$ → (F.V. formula of annuity due as the payment is made at the beg.)

$$R = \frac{F.V}{S_{\overline{n+1}|i}}$$

$$= \frac{10,00,000}{S_{\overline{10+1}|.12}} = \frac{10,00,000}{20.65458328}$$

$$\boxed{R = ₹\,50,878.72}$$

Questions for Mathematics of Finance

Ques:1. A ₹1000 bond paying annual dividends at the 8.5% will be redeemed at par at the end of 10 years. Find the purchase price of this bond if the investor wishes a yield rate of 8%.

Ques:2. Mr. X wants ₹ 5,00,000 at the end of 7 years. If the rate of interest is 8%, what amount shall be deposited at the beginning of each quarter so as to get the above amount?

Ques: 3. Mr. X purchased an asset for ₹ 1,00,000 on installment basis. Each installment is to repaid at the beginning of each quarter. Find the size of each installment if the money is to be repaid in 3 years and the rate of interest is 6% compounded quarterly.

Ques: 4. X buys a piece of land for which he agrees to make 10 annual payments of ₹20, 000 each, the first being made at the end of 3 years. Find the equivalent cash price of this property if the money is worth 5% effective.

Ques: 5. A house sells for ₹ 50,000 down and 10 semiannual payments of ₹ 5000 each, the first due 3 years hence. Find the cash price of the house if the money is worth 6% compounded semiannually.

Ques: 6. How much is needed to endure a series of lectures costing ₹2500 at the beginning of each quarter of each year indefinitely, if the money is worth 3% compounded annually?

Ques: 7. Suppose a machine costing ₹70, 000 is to be replaced at the end of 5 years, at what time it will have a salvage value of ₹10,000. In order to provide money at that time for a new machine costing the same amount, sinking fund is set up. The amount in the fund at that time is to be the difference between the replacement costs and salvage value. If equal payments are placed in the fund at the end of each quarter and the fund earns 8% compounded quarterly, what each payment be?

Ques: 8. A machine costing ₹52, 000 and its effective life is estimated to be 12 years. A sinking fund is created for replacing the machine by a new model at the end of its life time, when its scrap value is realized a sum of ₹5000 only. The price of a new model is estimated to be 25% higher than the price of a present one. Find what amount should be set aside at the end of each year, out of profits, for the sinking fund, if it accumulates at 10% effective.

Ques: 9. An income stream decreases continuously over time for x years, the income rate at t years from now being ₹ae^{-bt} per year. What is its present value if interest be reckoned at 100r% compounded continuously? Show that this equals the capital value of a uniform income stream of ₹ **a** per year for x years if the rate of interest is raised to 100(r + b)% per year.