Supplementary Study Material

Course: B.A (Hons.)Economics

Paper Name:Elements of Analysis

Semester: IV

Contents: Unit III:

Differentiation & Integration of Power Series

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Let the power series $\sum\limits_{n=0}^{\infty} a_n x^n$ have the radius of convergence $R > 0$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x$$
$$+ a_2 x^2 + \ldots, \ \ |x| < R.$$

Then for $|x| < R$ the function $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ is continuous. The power series can be differentiated term-by-term inside the interval of convergence. The derivative of the power series exists and is given by the formula

$$f'(x) = \frac{d}{dx} a_0 + \frac{d}{dx} a_1 x$$
$$+ \frac{d}{dx} a_2 x^2 + \ldots$$
$$= a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$
$$= \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

The power series can be also integrated term-by-term on an interval lying inside the interval of convergence. Hence, if $-R < b < x < R$, the following expression is valid:

$$\int_b^x f(t)\,dt = \int_b^x a_0 dt + \int_b^x a_1 t dt$$
$$+ \int_b^x a_2 t^2 dt + \ldots + \int_b^x a_n t^n dt + \ldots$$

If the series is integrated on the interval $[0, x]$, we can write:

$$\int_0^x f(t)\,dt = \int_0^x a_0 dt + \int_0^x a_1 t dt$$
$$+ \int_0^x a_2 t^2 dt + \ldots + \int_0^x a_n t^n dt + \ldots$$
$$= a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \ldots$$
$$= \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} + C.$$

Example 1.

Show that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

for $|x| < 1$.

Solution.

First we consider the power series:

$$1 + x + x^2 + x^3 + \ldots$$

This is a geometric series with ratio x. Therefore, it converges for $|x| < 1$. The sum of the series is $\frac{1}{1-x}$. Substituting $-x$ for x, we have

$$1 - x + x^2 - x^3 + \ldots = \frac{1}{1-(-x)}$$

$$= \frac{1}{1+x} \text{ for } |x| < 1.$$

Thus,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n \text{ for } |x| < 1.$$

Example 2.

Find a power series for the rational fraction $\frac{1}{2-x}$.

Solution.

We can write this function as

$$\frac{1}{2-x} = \frac{\frac{1}{2}}{1-\frac{x}{2}}.$$

As you can see, this is the sum of the infinite geometric series with the first term $\frac{1}{2}$ and ratio $\frac{x}{2}$:

$$\frac{1}{2} + \frac{1}{2}\frac{x}{2} + \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1}{2}\left(\frac{x}{2}\right)^3 + \ldots$$
$$= \frac{1}{2} + \frac{x}{2^2} + \frac{x^2}{2^3} + \frac{x^3}{2^4} + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}.$$

The given power series converges for $|x| < 2$.

Example 3.

Find a power series for $\frac{6x}{5x^2-4x-1}$.

Solution.

First we find the partial fraction decomposition for this function. The quadratic function in the denominator can be written as $5x^2-4x-1$
$=(5x+1)(x-1)$, so we can set:

$$\frac{6x}{5x^2-4x-1}=\frac{A}{5x+1}+\frac{B}{x-1}.$$

Multiply both sides of the expression by $5x^2-4x-1=(5x+1)(x-1)$ to obtain

$6x=A(x-1)+B(5x+1)$,
$\Rightarrow 6x=Ax-A+5Bx+B,$
$\Rightarrow 6x=(A+5B)x+(-A+B)$,
$$\Rightarrow \begin{cases} A+5B=6 \\ -A+B=0 \end{cases}.$$

The solution of this system of equations is $A = 1, B = 1$. Hence, the partial decomposition of the given function is

$$\frac{6x}{5x^2 - 4x - 1} = \frac{1}{5x+1} + \frac{1}{x-1}$$
$$= \frac{1}{1+5x} - \frac{1}{1-x}.$$

Both fractions are the sums of the infinite geometric series:

$$\frac{1}{1+5x} = \frac{1}{1-(-5x)}$$
$$= 1 - 5x + (-5x)^2 + (-5x)^3 + \dots$$
$$= \sum_{n=0}^{\infty} (-5x)^n,$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$
$$= \sum_{n=0}^{\infty} x^n.$$

Hence, the power series expansion of the initial function is

$$\frac{6x}{5x^2-4x-1} = \sum_{n=0}^{\infty}(-5x)^n - \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty}\left[(-5x)^n - x^n\right]$$

$$= \sum_{n=0}^{\infty}\left[(-5)^n - 1\right]x^n.$$

Example 4.

Find a power series representation for the function $\ln(1+x),\ |x|<1.$

Solution.

In Example 1 we found the power series expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

$$= \sum_{n=0}^{\infty}(-1)^n x^n,\ \ |x|<1.$$

Integrating this series term-by-term on the interval $[0, x]$, we find that

$$\ln(1+x) = \int_0^x \frac{dt}{1+t}$$

$$= \int_0^x \left(1 - t + t^2 - t^3 + \ldots\right) dt$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Example 5.

Represent the integral $\int_0^x \frac{\ln(1+t)}{t} dt$ as a power series expansion.

Solution.

In the previous problem (Example 4) we have found the power series expansion for logarithmic function:

$$\ln(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} t^n}{n}$$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots, \quad |t| < 1.$$

Then we can write:

$$\frac{\ln(1+t)}{t} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} t^{n-1}}{n}$$

$$= 1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \ldots$$

Integrating this series term-by-term on the interval $[0, x]$, we obtain

$$\int_0^x \frac{\ln(1+t)}{t}dt$$

$$= \int_0^x \left[1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \ldots\right] dt$$

$$= x - \frac{x^2}{2\cdot 2} + \frac{x^3}{3\cdot 3} - \frac{x^4}{4\cdot 4} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n^2}.$$

Example 6.

Obtain a power series representation for the exponential function e^x.

Solution.

Consider the series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!}$$
$$+ \frac{x^3}{3!} + \ldots,$$

that converges for all x.

Differentiating it term-by-term, we have

$$f'(x) = \frac{d}{dx}1 + \frac{d}{dx}x + \frac{d}{dx}\frac{x^2}{2!}$$
$$+ \frac{d}{dx}\frac{x^3}{3!} + \ldots = 0 + 1 + x + \frac{x^2}{2!} + \ldots$$
$$= f(x).$$

Hence, the function $f(x)$ satisfies the differential equation $f' = f$. The general solution of this equation has the form $f(x) = ce^x$, where c is a constant. Substituting the initial value $f(0) = 1$, we find that $c = 1$. Thus, we obtain the following power series expansion for e^x :

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Example 7.

Find a power series expansion for the hyperbolic sine function $\sinh x$.

In the previous example we obtained the formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Substituting $-x$ for x, we get

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$$

Then the expansion for the hyperbolic sine function has the form:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right.$$

$$\left.- \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right]$$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\right.$$

$$\left.- \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)\right]$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right.$$

$$\left. - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right]$$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\right.$$

$$\left. - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)\right]$$

$$= \frac{1}{2}\left[2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)\right]$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$