# Initial value Problems

Consider the following equation

$$f'(x) = -(x-1)^2$$

We want to find the $f(x)$ function and draw its graph in the $xy$ plane.

Using the property of Integral.

$$\int f'(x)\,dx = f(x) + C.$$

So. $\int f'(x)\,dx = \int -(x-1)^2\,dx.$

$$f(x) = -\frac{(x-1)^3}{3} + C.$$

The function $f(x)$ depends on the value $C$. Depending on what value $C$ takes, we get different graphs of $f(x)$ function

Eg if $C=0$; $f(x) = -\frac{(x-1)^3}{3}$

$C=1$; $f(x) = -\frac{(x-1)^3}{3} + 1$

$C=-1$; $f(x) = -\frac{(x-1)^3}{3} - 1$

and so on.

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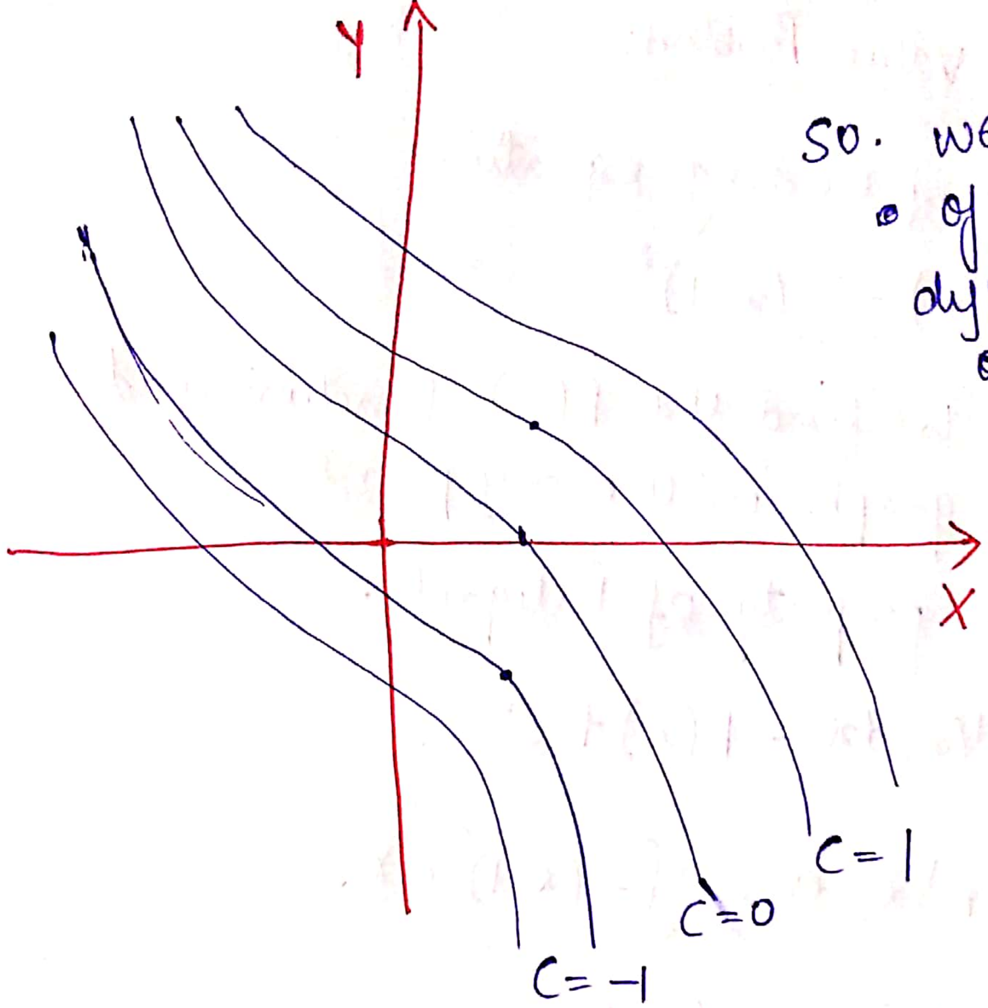


So. we have family of graphs for different values of C of F(x) function.

If we want the graph to pass through a specific point $(x_0, y_0)$; then the unique value of C will be determined and we will get the unique curve.

Eg Along with $F'(x) = -(x-1)^2$ we also know $F(1) = 1$ i.e. graph of $F(x)$ $f^n$ passes through $(1,1)$ then using

$$F(x) = -\frac{(x-1)^3}{3} + C$$

$$F(1) = -\frac{(1-1)^3}{3} + C = 1 \Rightarrow C = 1$$

So $F(x) = -\frac{(x-1)^3}{3} + 1$ is the required function.

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# Applications of Initial Value Problem

## ① Derivation of cost function from a Given Marginal cost functions

Marginal cost $MC = \frac{dC}{dx} = C'(x)$ where $C(x)$ is the total cost function.

$$\int MC\,dx = \int C'(x)dx = C(x) + k$$

k can be calculated with the help of fixed cost (i.e. the cost when no units are produced) or the cost of production of a specific no. of units of the commodity.

Average cost function $AC = \frac{C(x)}{x}$

Eg → $MC(x) = 3000e^{0.3x} + 100$

$FC = 60,000.$

$TC = C(x) = \int (3000\,e^{0.3x} + 100)\,dx$

$C(x) = 10000\,e^{0.3x} + 100x + k$

$C(0) = 60,000 \Rightarrow k = 60,000$

$C(x) = 10,000\,e^{0.3x} + 100x + 60,000$

$$AC = \frac{C(x)}{x} = \frac{10,000\,e^{0.3x} + 100x + 60,000}{x}$$

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## ② Derivation of Total Revenue function and inverse demand function from a given Marginal Revenue function.

$MR = \frac{dR}{dx}$ where $R = R(x)$ is the total Revenue function

$$R(x) = \int MR\, dx + k.$$

Since $R(0) = 0$, we can use this to calculate the value of $k$.

$AR = \frac{R}{x} = p$ (price) → inverse demand function.

<u>Example</u> $MR = \frac{6}{(x+2)^2} + 5$

$$TR = R(x) = \int \left(\frac{6}{(x+2)^2} + 5\right) dx$$

$$R(x) = -\frac{6}{x+2} + 5x + k$$

$$R(0) = 0 \Rightarrow k = 3.$$

So. $R(x) = \frac{-6}{x+2} + 5x + 3 = 5x + \frac{3x}{x+2}$

$$AR = \frac{R(x)}{x} = 5 + \frac{3}{x+2} = p$$

So. inverse demand $f^n$ $p = 5 + \frac{3}{x+2}$

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## (3) Determination of profit function if Marginal Revenue and Marginal cost functions are given

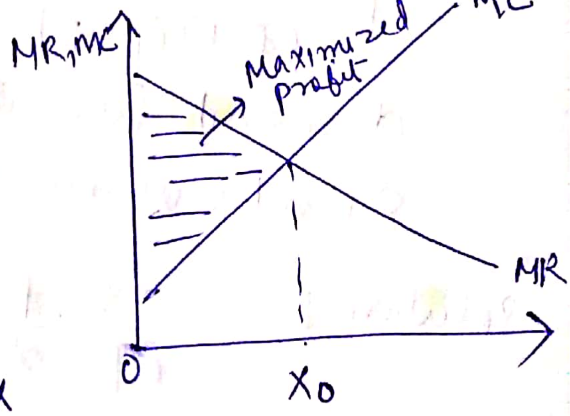
$$\frac{d\pi}{dx} = MR - MC$$

$$\pi(x) = \int \frac{d\pi}{dx} = \int (MR - MC)\, dx + k$$

$k$ can be evaluated by using either FC or loss at zero level of output.

Point where profit is maximized is $MR = MC$ [let say $x_0$]

So. maximized profit $= \int_0^{x_0} (MR - MC)\, dx$




Example: $MC = 300 + \frac{x}{20}$

$MR = 400$

$FC = 5000$

$$\pi(x) = \int \left(400 - \left(300 + x/20\right)\right) dx$$

$$\pi(x) = 100x - \frac{x^2}{40} + k. \qquad \pi(0) = -5000.$$

So. $\boxed{\pi(x) = 100x - \frac{x^2}{40} - 5000}$ → Profit fn.

Profit maximization → $MR = MC \Rightarrow 400 = 300 + x/20$

$x^* = 2000.$

Maximized profit $= \int_0^{2000} \left(100 - \frac{x}{20}\right) dx$ or

$$\pi^*(2000) = 100x^* - \frac{(x^*)^2}{40} - 5000 = 95{,}000.$$

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## (4) Derivation of Demand function from a Given price elasticity of demand

$$e_{yx} = \frac{dy}{dx} \cdot \frac{x}{y} \quad \text{or} \quad e_{xp} = \frac{dx}{dp} \cdot \frac{p}{x}$$

$$e_{xp} \frac{dp}{p} = \frac{dx}{x}$$

Integrating both sides we get $x(p)$ which is the demand $f^n$.

**Example:** Let price elasticity of demand $e_{xp} = \frac{-3p}{(p-1)(p+2)}$. Find the corresponding demand function if quantity demanded is 8 units when price is 2.

**Solution:**

$$\frac{dx}{dp} \cdot \frac{p}{x} = \frac{-3p}{(p-1)(p+2)}$$

$$\frac{dx}{x} = \frac{-3p}{(p-1)(p+2)} \frac{dp}{p} = \frac{-3}{(p-1)(p+2)} dp.$$

$$\frac{dx}{x} = \left(\frac{1}{p+2} - \frac{1}{p-1}\right) dp$$

$$\int \frac{dx}{x} = \int \left(\frac{1}{p+2} - \frac{1}{p-1}\right) dp$$

$$\log_e x = \log_e(p+2) - \log(p-1) + k_2 \quad [\text{or } \log_e k]$$

$$\log_e x = \log_e \left(\frac{(p+2)k}{p-1}\right)$$

$$x = \left(\frac{p+2}{p-1}\right) k \Rightarrow \text{when } p = 2 \quad x = 8 \Rightarrow k = 2$$

So $\boxed{x(p) = 2(p+2)/(p-1)}$

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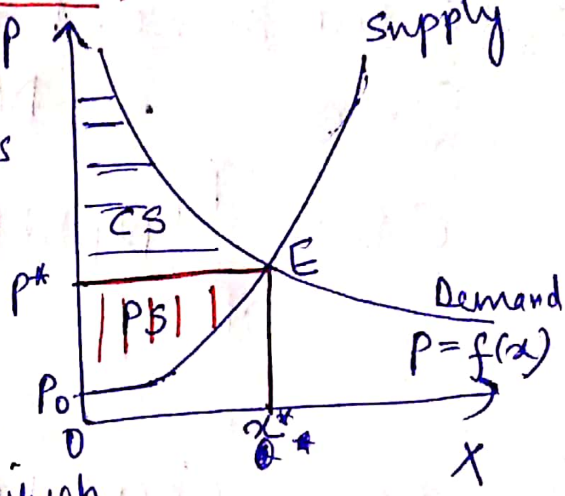
# (5) Consumer and Producer Surplus.




**Consumer's Surplus** is the difference between what consumers are willing to pay and what they actually pay.

**Producer Surplus** is the difference between the actual price at which the quantity is supplied and the minimum price at which producers are willing to supply it.

If the inverse demand fn and inverse supply function is given

$P = f(x) \rightarrow$ inverse dd. $\quad P = g(x) \rightarrow$ inverse supply

CS = Total area under inverse demand curve from 0 to $x^*$ minus the area of rectangle $Ox^*EP^*$

$$= \int_0^{x^*} f(x)\,dx - x^* p^*.$$

PS = Area of rectangle $Ox^*Ep^*$ minus total area under the inverse supply curve from 0 to $x^*$

$$P = x^* p^* - \int_0^{x^*} g(x)\,dx$$

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If the demand function and supply $f^n$ is given $x = D(p)$ and $x = S(p)$

$$CS = \int_{p^*}^{\bar{P}} D(p)\,dP$$

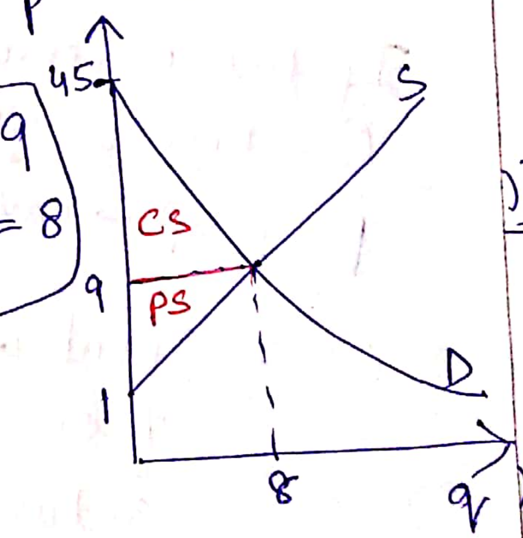
where $\bar{P}$ is a price when $x=0$ [or max possible price] ~~of~~ ~~where~~ for demand $f^n$

$$PS = \int_{P_0}^{p^*} S(P)\,dP$$

~~If~~ where $P_0$ is a price when $x=0$ in supply $f^n$.

**Example:** Let ~~the~~ demand $f^n$ is $q = \frac{90}{p} - 2$

and supply $f^n$ $q = p-1$




Eq$^m$ $\frac{90}{p} - 2 = p - 1 \Rightarrow p^* = 9$, $q^* = 8$

$$CS = \int_9^{45} \left(\frac{90}{P} - 2\right) dP$$

$$= \int_0^8 \left(\frac{90}{q+2}\right) dq - 9 \times 8$$

$$= 90 \log_e 5 - 72$$

$$PS = \int_1^9 (p-1)\,dp = \int 9\times 8 - \int_0^8 (q+1)\,dq$$

$$= 32$$

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# Economic Applications of Integration

(Reference section 10.4 Sydsaeter & Hammond).

## (1) Extraction from an Oil Well

Let at time $t=0$, extraction of oil starts from a well that contains $K$ barrels of oil.

Let $x(t)$ = amount of oil in barrels that is left at time $t$

$x(t)$ is a decreasing function of $t$ and $x(0) = K$.

Amount of oil that is extracted in a time interval $[t, t+\Delta t]$ is $x(t) - x(t+\Delta t)$ $[\Delta t > 0]$

Extraction per unit of time is $\frac{x(t)-x(t+\Delta t)}{\Delta t} = -\frac{(x(t+\Delta t)-x(t))}{\Delta t}$

Let $x(t)$ is a differentiable $f^n$ of time

So as $\Delta t \to 0$ $\quad \frac{x(t+\Delta t)-x(t)}{\Delta (t)} \to \frac{dx}{dt}$ or $\dot{x}(t)$

Define $u(t)$ be rate of extraction at time $t$.

So $\quad u(t) = \lim_{\Delta t \to 0} \frac{x(t)-x(t+\Delta t)}{\Delta (t)} = -\dot{x}(t)$

or $\boxed{\dot{x}(t) = -u(t) \quad \text{with } x(0) = K}$.

So we have Initial value problem now.

Integrate both sides $\int_0^t \dot{x}(t)\,dt = \int_0^t -u(s)\,ds$.

$$x(t) - x(0) = -\int_0^t u(s)\,ds \quad [\text{s being the variable of integration}]$$

$$x(t) = K - \int_0^t u(s)\,ds \rightarrow \text{Solution.}$$

So. The amount of oil left at time $t$ is equal to the initial amount $K$ minus the total amount that has been extracted during the time span $[0, t]$

- If rate of extraction is constant i.e. $u(t) = \bar{u}$. Then

$$x(t) = K - \int_0^t \bar{u}\,ds = K - \bar{u}[s]_0^t = K - \bar{u}t$$

The well will be empty with $x(t) = 0$.

or $K - \bar{u}t = 0$ or $t = \dfrac{K}{\bar{u}}$

<u>Note:</u> $x(t)$ is a stock variable
$u(t)$ is a flow variable

<u>Question:</u> Let $u(t) = \bar{u}e^{-at}$ $(a > 0)$ and $x(0) = x_0$. Find an expression $x(t)$ for the remaining amount of oil at time $t$. Under what conditions will the well never be exhausted?

<u>Solution:</u> $x(t) = x_0 - \int_0^t \bar{u}e^{-as}\,ds = x_0 - \dfrac{\bar{u}}{a} + \dfrac{\bar{u}}{a}e^{-at}$

as $t \to \infty$ $x(t) \to x_0 - \dfrac{\bar{u}}{a}$, so if $x_0 > \bar{u}/a$, $x(t) > 0$
so well will never be exhausted.

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## 2. Present Discounted value of a Continuous Future Income Stream

Recall : When series of future payments are made at specific discrete moments in time .. Then their present value will be .

Let $t=1$ $t=2$ - - - $t=n$

$a_1$ $a_2$ $a_n$.

(with annual compounding) $PV = \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_n}{(1+r)^n}$

(Half yearly compounding) $PV = \frac{a_1}{(1+r/2)^2} + \frac{a_2}{(1+r/2)^4} + \cdots + \frac{a_n}{(1+r/2)^{2n}}$

and so on

(continuous compounding) $PV = a_1 e^{-r} + a_2 e^{-2r} + \cdots + a_n e^{-nr}$

Now let that income is to be received continuously from time $t=0$ to time $t=T$ at the rate of $f(t)$ dollars per year at time $t$.

Let interest is compounded continuously at rate $r$.

Let $P(t)$ be the present discounted value of all the payments made over the interval $[0,t]$.

If $dt$ is any number, the present value of the income received in the interval $[t, t+dt]$

is $P(t+dt) - P(t)$. If $dt$ is a small number, the income received in this interval is approximately $f(t)\,dt$ and PDV of this amount is approximately $f(t)e^{-rt}dt$

Thus $P(t+dt) - P(t) \approx f(t)e^{-rt}dt$

or $$\frac{P(t+dt) - P(t)}{dt} \approx f(t)e^{-rt}$$

as $dt \to 0$ $\quad P'(t) = f(t)e^{-rt}$

By definition of definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ where } F'(x) = f(x)$$

So, $$P(T) - P(0) = \int_0^T f(t)e^{-rt}dt$$

as $P(0) = 0$.

The Present discounted value (at time 0) of a continuous income stream at the rate of $f(t)$ dollars per year over the interval $[0, T]$ with continuously compounded interest at rate $r$ is given by

$$PDV = \int_0^T f(t)e^{-rt}dt \quad -(1)$$

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Equation ① gives the value at time 0 of income stream $f(t)$ received during time interval $[0,T]$

Value at time T will be $e^{(rT)}\left[\int_0^T f(t)e^{-rt}\,dt\right]$.

or $\int_0^T f(t)\,e^{r(T-t)}\,dt$ as $e^{rT}$ is a constant

Future discounted value (at time T) of continuous income stream $(FDV) = \int_0^T f(t)\,e^{r(T-t)}\,dt$

Discounted value at any time $s \in [0,T]$ of an income stream $f(t)$ received during time interval $[s,T]$ is $\cancel{e^{rs}\int_0^T f(t)e^{-rt}\,dt}$

$$\int_{t=s}^{T} f(t)\,e^{-r(t-s)}\,dt$$

Question:- Find PDV and FDV of a constant income stream of \$1000 per year over the next 10 years, assuming an interest rate of $r = 8\% = 0.08$ annually compounded continuously

Solution $PDV = \int_0^{10} 1000\,e^{-0.08t}\,dt = 1000\left(\dfrac{e^{-0.08t}}{-0.08}\right)_0^{10}$

$= \dfrac{1000}{0.08}\left(1-e^{-0.8}\right) \approx 6883.39$

$FDV = e^{0.08\times 10}\,PDV \approx e^{0.8} \times 6883.39 \approx 15319.27$

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# ③ Income Distribution

Consider an economy A with $n$ individuals in the population. Let the income of individuals be measured in dollars. Let $r_0$ is the lowest and $r_1$ is the highest income in the group.

- $F(r)$ denote the proportion of individuals who receive no more than $r$ dollars.

So $nF(r)$ is the number of individuals with income no greater than $r$.

$$r_0 \leq r \leq r_1$$

- Since $r$ has to be a multiple of \$ 0.01 (income measured in dollars) and $F(r)$ has to be a multiple of $1/n$ [as it is proportion]

By definition, $F$ is not a continuous and differentiable $f^n$ in the interval $[r_0, r_1]$.

However if $n$ is large enough, it is usually possible to find a "smooth" function that gives a good approximation to the true income distribution.

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Assume that $F$ is a function with continuous partial derivative $f$

$$f(r) = F'(r) \quad \forall\ r \in (r_0, r_1)$$

or $F(r+\Delta r) - F(r) \approx f(r)\,\Delta r$ [linear approximation]

(for small $\Delta r$)

$f(r)\,\Delta r$ represents approximately the proportion of individuals who earn between $r$ and $r+\Delta r$.

- function $f$ is called an income density function
  $F$ is the associated cumulative distribution function

Let $f$ is a continuous income distribution for a certain population with incomes in the interval $[r_0, r_1]$.

Let $r_0 \le a \le b \le r_1$, $\int_a^b f(r)\,dr$ is the proportion of individuals with incomes in $[a,b]$

→ $n \int_a^b f(r)\,dr$ = number of individuals with incomes in the interval $[a, b]$

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To find out combined income of those who earn between a and b dollars.

Let $M(r)$ denote the total income of those who earn no more than $r$ dollars or between $[r_0, r]$

Consider the interval $[r, r+\Delta r]$

$M(r+\Delta r) - M(r)$ is the total income of those who have income in the interval $[r, r+\Delta r]$ $[\Delta r > 0]$.

In this interval, there are $n[F(r+\Delta r) - F(r)]$ individuals each of whom earns at most $r+\Delta r$ and atleast $r$.

Thus

$$\underbrace{nr[F(r+\Delta r) - F(r)]}_{\text{This is the total income when all indivials earn exactly } r} \le M(r+\Delta r) - M(r) \le \underbrace{n(r+\Delta r)[F(r+\Delta r) - F(r)]}_{\text{This is the total income when all individuals earn exactly } r+\Delta r}$$

Divide by $\Delta r$

$$\frac{nr[F(r+\Delta r) - F(r)]}{\Delta r} \le \frac{M(r+\Delta r) - M(r)}{\Delta r} \le \frac{n(r+\Delta r)[F(r+\Delta r) - F(r)]}{\Delta r}$$

$\Delta r \to 0$ $\quad nrF'(r) \le M'(r) \le nrF'(r)$

$$\Rightarrow M'(r) = nrF'(r) = nrf(r)$$

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Total income of individuals with incomes in the interval [a, b] is M(b) − M(a)

Since $M'(r) = nr f(r)$

So $\int_a^b M'(r)\,dr = \int_a^b nr f(r)\,dr$.

$$\boxed{M(b) - M(a) = n\int_a^b r f(r)\,dr}$$

Mean income of individuals with incomes in the interval [a, b] = $\dfrac{\text{Total income of individuals with incomes in the interval } [a,b]}{\text{Number of individuals with incomes in the interval } [a,b]}$

$$= \frac{n\int_a^b r f(r)\,dr}{n\int_a^b f(r)\,dr} = \frac{\int_a^b r f(r)\,dr}{\int_a^b f(r)\,dr}.$$

Example: Let $f(r) = Br^{-2}$. Find the mean income $m$ over the interval $[b, 2b]$

Solution No. of individuals in the interval $[b, 2b]$

$$= n\int_a^b f(r)\,dr = n\int_b^{2b} Br^{-2}\,dr = nB\left[\frac{r^{-1}}{-1}\right]_b^{2b}$$

$$= \frac{nB}{2b}$$

Total income of individuals with incomes in the interval $[b, 2b]$ = $n\int_b^{2b} r f(r)\,dr = n\int_b^{2b} r\,Br^{-2}\,dr$

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$$= n\int_b^{2b} B r^{-1} dr = nB\left[\log|r|\right]_b^{2b}$$

$$= nB[\log 2b - \log b]$$

$$= nB\log 2$$

$$\text{Mean income} = \frac{nB\log 2}{nB/2b} = 2b\log_e 2$$

## Influence of Income distribution on demand

Let $D(p, r)$ be a continuous function that denotes the number of commodity units demanded by an individual with income $r$ when the price per unit is $p$.

Let $f(r)$ is the income distribution [as before] and $a \le r \le b$.

We need to find the total demand for the commodity when the price is $p$.

Methodology remains the same as done in previous applications

Let $T(r)$ be the total demand for the commodity by all individuals who earn less than or equal to $r$ [when $p$ is fixed]

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Consider the interval $[r, r+\Delta r]$.

$T(r+\Delta r)-T(r) \rightarrow$ Total demand of individuals with incomes in the interval $[r, r+\Delta r]$

$n[F(r+\Delta r)-F(r)] \rightarrow$ Total no. of individuals with income in this interval

$D(p, r)$ is the demand of an individual when income is $r$.

So. $nD(p,r)[F(r+\Delta r)-F(r)] \rightarrow$ Total demand of individuals if all have ~~with~~ income $r$ in this interval

Similarly $nD(P, r+\Delta r)[F(r+\Delta r)-F(r)]$ is the total demand if all ~~order~~ individuals who have income in the interval $[r, r+\Delta r]$ have income $r+\Delta r$.

So. $nD(P,r)[F(r+\Delta r)-F(r)] \leq T(r+\Delta r)-T(r) \leq nD(P, r+\Delta r)[F(r+\Delta r)-F(r)]$

Now dividing by $\Delta r$ and taking the limit $\Delta r \to 0$ we get

$$T'(r) = nD(P,r)F'(r) = nD(P,r)f(r)$$

or $$T(b)-T(a) = \int_a^b nD(P,r)f(r)\,dr$$

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Total demand for the commodity when the price is $p$ and $a \le r \le b$; denoted by $x(p)$ is

$$\boxed{x(p) = \int_a^b n D(p,r) f(r)\, dr}$$

Example : $f(r) = Br^{-2}$ Interval $[b, 2b]$

$D(p,r) = A p^{\alpha} r^{\nu}$ $A > 0$; $\alpha < 0$, $\nu > 0$, & $\nu \neq 1$

Let there are $n$ individuals in the population

Total demand $x(p) = \int_b^{2b} n (A p^{\alpha} r^{\nu})(Br^{-2})\, dr$

$$x(p) = \int_b^{2b} n A B p^{\alpha} r^{\nu-2}\, dr$$

$$x(p) = nABp^{\alpha}\left[\int_b^{2b} r^{\nu-2}\, dr\right]$$

$$x(p) = nABp^{\alpha}\left[\frac{r^{\nu-1}}{\nu-1}\right]_b^{2b}$$

$$x(p) = \frac{nABp^{\alpha}}{\nu-1}\left[(2b)^{\nu-1} - [b]^{\nu-1}\right]$$

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