GE IV: ELEMENTS OF ANALYSIS
B.A(HONS.) ECONOMICS
SEMESTER IV

SUPPLEMENTARY STUDY MATERIAL
UNIT II: INFINITE SERIES-(REVISION)

INFINITE - SERIES

Basic defns & Necessary condition

DEFN: An infinite series is an infinite sum

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

DEFN Let $\sum u_n$ be an infinite series

Define its Sequence of Partial Sums (S.O.P.S) $\langle S_n \rangle$ as

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$\vdots$$
$$S_n = u_1 + u_2 + \cdots + u_n$$
$$\vdots$$

DEFN An infinite series $\sum u_n$ is said to converge/diverge/oscillate according as its S.O.P.S $\langle S_n \rangle$ converges/diverges/oscillates resp.

Also sum of series = limit of its S.O.P.S.

Necessary Condition for convergence

A necessary condition for convergence of $\sum u_n$ is $\lim_{n\to\infty} u_n = 0$.

Proof: Let $\sum u_n$ converge

ie, its SOPS $\langle S_n \rangle$ converges, to say l.

where
$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$\vdots$$
$$S_{n-1} = u_1 + u_2 + \cdots + u_{n-1}$$
$$S_n = u_1 + u_2 + \cdots + u_n$$

Clearly, $u_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$
$$= l - l$$
$$= 0.$$

Show that the foll. series do not converge:

(i) $\sum \frac{n}{n+1}$ (ii) $\sum n^{1/n}$

(iii) $\sum \sin\frac{n\pi}{3}$ (iv) $\sum (-1)^n$

Soln (i) $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ $\therefore \sum u_n$ does not conv.

(ii) $\lim_{n\to\infty} n^{1/n} = 1 \neq 0$ $\therefore \sum u_n$ does not conv.

(iii) $\lim_{n\to\infty} \sin\frac{n\pi}{3}$ does not exist $\therefore \sum u_n$ does not conv.

(iv) $\lim_{n\to\infty} (-1)^n$ does not exist $\therefore \sum u_n$ does not conv

SERIES OF NON-NEGATIVE TERMS ②

Henceforth, let $\sum u_n$ be a series of non-negative terms, ie $u_n \geq 0 \; \forall n \in$

Then clearly its S.O.P.S $\langle S_n \rangle$ is $\uparrow$. Hence:

1) $\langle S_n \rangle$ cannot oscillate; it either converges or diverges to ∞.

2) A nec. & suff condition for $\langle S_n \rangle$ to converge is that it is bounded above.

BASIC COMPARISON TEST (B.C.T.)

Let $\sum u_n$ & $\sum v_n$ be 2 series of non-neg. terms.

Suppose that $\exists \; K > 0$ & $n_0 \in N$ st:

$$u_n \leq K v_n \qquad \forall n \geq n_0.$$

If $\sum v_n$ converges, then $\sum u_n$ also converges

Taking contrapositive, we have:

$\sum u_n$ does not converge $\Rightarrow$ $\sum v_n$ does not converge

ie, $\sum u_n$ diverges $\Rightarrow$ $\sum v_n$ diverges [$\because$ series of non-neg terms cannot oscillate]

Two Companion Series

(A) Geometric Series $\sum ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$

Result (Geometric Series) The geometric series $\sum ar^{n-1}$ $(a>0, r>0)$ converges if $r<1$ & diverges if $r \geq 1$.

Proof Case (i) $r < 1$

Let $\langle S_n \rangle$ be the S.O.P.S of $\sum ar^{n-1}$

ie, $S_n = a + ar + ar^2 + \cdots\cdots + ar^{n-1}$

$$= \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r} - \frac{ar^n}{1-r}$$

$$\leq \frac{a}{1-r} \quad [\because 1-r>0]$$

$\therefore \langle S_n \rangle$ is bounded above

Also, $\langle S_n \rangle$ is $\uparrow$ $[\because ar^{n-1} \geq 0 \ \forall n]$ (By M.C.T)

$\therefore \langle S_n \rangle$ converges, ie $\sum ar^{n-1}$ converges.

Case (ii) $r = 1$

Then $\sum ar^{n-1} = a + a + a + \cdots\cdots$

The S.O.P.S $\langle S_n \rangle$ is $\langle a, 2a, 3a, \cdots\cdots \rangle \to \infty$

$\therefore \sum ar^{n-1}$ diverges.

Case (iii) $r > 1$

The series is $\sum ar^{n-1} = a + ar + ar^2 + \cdots\cdots$ — (1) $(r>1)$

Consider the series: $a + a + a + \cdots\cdots$ — (2)

Each term of (1) is $\geq$ corresp. term of (2) $[\because r>1]$

Also, series (2) diverges [by case (ii)]

$\therefore$ series (1) also diverges [by B.C.T]

(5) The "p-Series"

$$\sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\cdots + \frac{1}{n^p} + \cdots\cdots$$

ARTICLE (p-Series) : The series $\sum \frac{1}{n^p}$ converges if $p > 1$ & diverges if $p \le 1$

PROOF: Case (i) $p > 1$

Let $\langle S_n \rangle$ be the S.O.P.S of $\sum \frac{1}{n^p}$

ie $S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\cdots\cdots\cdots + \frac{1}{n^p}$

$\boxed{\text{Choose } k \in \mathbb{N} \text{ s.t. } n \le 2^k - 1}$

Then
$$S_n \le 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \cdots + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right) + \cdots\cdots + \left(\frac{1}{(2^{k-1})^p} + \cdots\cdots + \frac{1}{(2^k-1)^p}\right)$$

$$\le 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \underbrace{\left(\frac{1}{4^p} + \cdots + \frac{1}{4^p}\right)}_{4 \text{ terms}} + \underbrace{\left(\frac{1}{8^p} + \cdots\cdots + \frac{1}{8^p}\right)}_{8 \text{ terms}} + \cdots\cdots + \underbrace{\left(\frac{1}{(2^{k-1})^p} + \frac{1}{(2^{k-1})^p} + \cdots\cdots + \frac{1}{(2^{k-1})^p}\right)}_{2^{k-1} \text{ terms}}$$

$$= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots\cdots + \frac{2^{k-1}}{(2^{k-1})^p}$$

$$= 1 + \frac{2}{2^p} + \left(\frac{2}{2^p}\right)^2 + \left(\frac{2}{2^p}\right)^3 + \cdots\cdots + \left(\frac{2}{2^p}\right)^{k-1}$$

$$= 1 + r + r^2 + r^3 + \cdots\cdots + r^{k-1}, \quad \text{where } r = \frac{2}{2^p} < 1 \quad [\because p > 1]$$

$$= \frac{1(1 - r^k)}{1 - r}$$

$$= \frac{1}{1-r} - \frac{r^k}{1-r}$$

$$\le \frac{1}{1-r} \qquad [\because 1 - r > 0]$$

$\therefore$ $\langle S_n \rangle$ is bounded above

Also, $\langle S_n \rangle$ is $\uparrow$ $\quad [\because \frac{1}{n^p} > 0 \ \forall n \in \mathbb{N}]$

$\therefore$ $\langle S_n \rangle$ converges, ie $\sum \frac{1}{n^p}$ converges

LIMIT-COMPARISON TEST (L.C.T.)

(5)

Let $\sum u_n$ & $\sum V_n$ be 2 series of positive terms

If $\lim\limits_{n \to \infty} \frac{u_n}{V_n}$ is finite & non-zero, then $\sum u_n$ & $\sum V_n$ either both converge or both diverge (ie, have same behaviour)

PROBLEM SET I (Comparison Test) (L.C.T.)

ex 1) Test for convergence:

$$\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \cdots\cdots$$

Soln $\quad u_n = \dfrac{\sqrt{n}}{2n+3} = \dfrac{\sqrt{n}}{n\left(2+\frac{3}{n}\right)} = \dfrac{1}{n^{1/2}\left(2+\frac{3}{n}\right)}$

Take $V_n = \dfrac{1}{n^{1/2}}$

$\dfrac{u_n}{V_n} = \dfrac{1}{2+\frac{3}{n}} \quad \therefore \lim\limits_{n \to \infty} \dfrac{u_n}{V_n} = \dfrac{1}{2}$ which is finite & non-zero

$\therefore$ by Comparison Test, $\sum u_n$ & $\sum V_n$ have same behaviour

Now $\sum V_n$ diverges (as $p = \frac{1}{2} < 1$)

$\therefore \sum u_n$ also diverges.

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ex2) Test for convergence:

$$\frac{1}{3.7} + \frac{1}{4.9} + \frac{1}{5.11} + \cdots\cdots$$

Soln $\quad u_n = \dfrac{1}{(n+2)(2n+5)} = \dfrac{1}{n^2\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)}$.

Take $V_n = \dfrac{1}{n^2}$

$\dfrac{u_n}{V_n} = \dfrac{1}{\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)} \quad \therefore \lim\limits_{n \to \infty} \dfrac{u_n}{V_n} = \dfrac{1}{1.2} = \dfrac{1}{2}$ which is finite & non-zero

$\therefore$ by Comparison Test, $\sum u_n$ & $\sum V_n$ have same behaviour.

Now $\sum V_n$ converges (as $p = 2 > 1$)

$\therefore \sum u_n$ also converges

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ex3) Test for convergence:

$$\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1}.$$

Soln $\quad u_n = \dfrac{\sqrt{n+1}-1}{(n+2)^3-1} = \dfrac{\sqrt{n}\left\{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}\right\}}{n^3\left\{\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}\right\}} = \dfrac{1}{n^{5/2}} \dfrac{\left\{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n^3}}\right\}}{\left\{\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}\right\}}$

Take $V_n = \dfrac{1}{n^{5/2}}$

$\therefore \dfrac{u_n}{V_n} = \dfrac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}} \quad \therefore \lim\limits_{n \to \infty} \dfrac{u_n}{V_n} = \dfrac{1-0}{1-0} = 1$ (finite & non-zero)

$\therefore$ by Comparison Test, $\sum u_n$ & $\sum V_n$ have same behaviours.

Now $\sum V_n$ converges (as $p = 5/2 > 1$)

$\sum u_n$ also converges

Based on Rationalising:

ex 1) Test for convergence : $\sum \left(\sqrt{n^3+1} - \sqrt{n^3}\right)$

Soln. $u_n = \sqrt{n^3+1} - \sqrt{n^3}$

$$= \left(\sqrt{n^3+1} - \sqrt{n^3}\right)\frac{\left(\sqrt{n^3+1} + \sqrt{n^3}\right)}{\left(\sqrt{n^3+1} + \sqrt{n^3}\right)}$$

$$= \frac{(n^3+1) - n^3}{\sqrt{n^3+1} + \sqrt{n^3}} = \frac{1}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \frac{1}{n^{3/2}\left\{\sqrt{1+\frac{1}{n^3}} + 1\right\}}$$

Take $v_n = \frac{1}{n^{3/2}}$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{1+\frac{1}{n^3}} + 1} \qquad \therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2}$$, which is finite & non-zero

$\therefore$ by Comparison Test, $\sum u_n$ & $\sum v_n$ have same behaviour

Now $\sum v_n$ converges $\left(\text{as } p = 3/2 > 1\right)$

$\therefore \sum u_n$ also converges.

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ex 2) Test for convergence : $\sum \left(\frac{\sqrt{n+1} - \sqrt{n-1}}{n}\right)$

Soln $u_n = \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$

$$= \left(\frac{\sqrt{n+1} - \sqrt{n-1}}{n}\right)\frac{\left(\sqrt{n+1} + \sqrt{n-1}\right)}{\left(\sqrt{n+1} + \sqrt{n-1}\right)}$$

$$= \frac{(n+1) - (n-1)}{n\left(\sqrt{n+1} + \sqrt{n-1}\right)} = \frac{2}{n^{3/2}\left\{\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}\right\}}$$

Take $v_n = \frac{1}{n^{3/2}}$

$$\frac{u_n}{v_n} = \frac{2}{\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}} \qquad \therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{2}{2} = 1$$ (finite & non-zero)

$\therefore$ by Comparison Test, $\sum u_n$ & $\sum v_n$ have same behaviour.

Now $\sum v_n$ converges $\left(\text{as } p = 3/2 > 1\right)$

$\therefore \sum u_n$ also converges

Trigonometric Problems (also done by Comparison Test)

ex 1) Test for convergence :

(a) $\sum \sin \frac{1}{n^2}$ (b) $\sum \sin \frac{1}{n}$ *(c) $\sum \frac{1}{\sqrt{n}} \sin \frac{1}{n}$

[Similar for Tan]

Soln (a) $u_n = \sin \frac{1}{n^2}$

Take $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{\sin \frac{1}{n^2}}{\frac{1}{n^2}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{x\to 0} \frac{\sin x}{x} \quad \left[\text{by putting } x = \frac{1}{n^2}\right]$$

$= 1$ (finite & non-zero)

$\therefore$ by Comparison Test, $\sum u_n$ & $\sum v_n$ have same behaviour.

Now $\sum v_n$ converges (as $p = 2 > 1$)

$\therefore$ $\sum u_n$ also converges

(b) $u_n = \sin \frac{1}{n}$ Do yourself Ans Divergent

(c) $u_n = \frac{1}{\sqrt{n}} \sin \frac{1}{n}$

(*) Take $v_n = \frac{1}{\sqrt{n}} \cdot \frac{1}{n} = \frac{1}{n^{3/2}}$ (Note !)

$$\frac{u_n}{v_n} = \frac{\frac{1}{\sqrt{n}} \sin \frac{1}{n}}{\frac{1}{\sqrt{n}} \cdot \frac{1}{n}} = \frac{\sin \frac{1}{n}}{\frac{1}{n}}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \text{ (finite \& non-zero)}$$

$\therefore$ by Comparison Test, $\sum u_n$ & $\sum v_n$ have same behaviour

Now $\sum v_n$ converges (as $p = 3/2 > 1$)

$\therefore$ $\sum u_n$ also converges.

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ex 2) Test for convergence :

(a) $\sum \cos \frac{1}{n^2}$ (b) $\sum \cos \frac{1}{n}$ (c) $\sum \frac{1}{\sqrt{n}} \cos \frac{1}{n}$

Soln (a) $u_n = \cos \frac{1}{n^2}$

$$\lim_{n\to\infty} u_n = \cos 0 = 1 \neq 0$$

$\therefore$ $\sum u_n$ does not converge

(b) & (c) Try Yourself.

CAUCHY'S ROOT TEST

CAUCHY'S ROOT TEST Let $\sum u_n$ be a series of positive terms.

Let $l = \lim_{n\to\infty} u_n^{1/n}$.

If $l < 1$, then $\sum u_n$ converges

If $l > 1$, " " diverges

If $l = 1$, " " may converge or diverge

PROBLEM SET II (Cauchy's Root Test)

[to be used when $u_n = (\quad)^n$]

ex 1) Test for convergence. $\frac{2}{3} + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{7}\right)^3 + \cdots\cdots$

Soln. $u_n = \left(\frac{n+1}{2n+1}\right)^n$

$$l = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{n+1}{2n+1} = \lim_{n\to\infty} \frac{n(1+\frac{1}{n})}{n(2+\frac{1}{n})} = \frac{1}{2} < 1$$

$\therefore$ by Root Test, $\sum u_n$ converges.

ex 2) Test for convergence : $\sum \left(n^{1/n} - 1\right)^n$

Soln $u_n = \left(n^{1/n} - 1\right)^n$

$$l = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(n^{1/n} - 1\right) = 1 - 1 = 0 < 1$$

$\therefore$ by Root Test, $\sum u_n$ converges

ex 3) Test for convergence: $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

Soln : $u_n = \frac{1}{(\log n)^n}$

$$l = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1$$

$\therefore$ by Root Test, $\sum u_n$ converges

** ex 4) Test for convergence : $\sum \left(\frac{n}{n-1}\right)^{n^2}$

Soln: $u_n = \left(\frac{n}{n-1}\right)^{n^2}$

$$l = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n-1}\right)^n = \lim_{n\to\infty} \frac{n^n}{n^n(1-\frac{1}{n})^n}$$

$$= \lim_{n\to\infty} \frac{1}{(1-\frac{1}{n})^n} = \frac{1}{e^{-1}} = e > 1$$

$\therefore$ by Root Test, $\sum u_n$ diverges

* ex 5) Test for convergence : $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Soln $u_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$$l = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{1/2}} = \lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$= \frac{1}{e} < 1$$

$\therefore$ by Root Test, $\sum u_n$ converges

D'ALEMBERT'S RATIO TEST

D'ALEMBERT'S RATIO TEST Let $\sum u_n$ be a series of positive terms

Let $l = \lim_{n \to \infty} \frac{u_n}{u_{n+1}}$

If $l > 1$, then $\sum u_n$ converges

If $l < 1$, " " diverges

If $l = 1$, " " may converge or diverge

D'ALEMBERT'S RATIO TEST

☆ 1st Version $\left(\frac{u_n}{u_{n+1}}\right)$

Let $\sum u_n$ be a series,

$u_n > 0 \quad \forall n \in \mathbb{N}$

Let $l = \lim_{n \to \infty} \frac{u_n}{u_{n+1}}$

If $l > 1$, then $\sum u_n$ converges

" $l < 1$, " $\sum u_n$ diverges

2nd Version $\left(\frac{u_{n+1}}{u_n}\right)$

Let $\sum u_n$ be a series,

$u_n > 0 \quad \forall n \in \mathbb{N}$

Let $l = \lim_{n \to \infty} \frac{u_{n+1}}{u_n}$

If $l < 1$, then $\sum u_n$ converges

" $l > 1$, " $\sum u_n$ diverges

PROBLEM SET III (Ratio Test)

[To be used when u_n involves x^n or $\lfloor n$ or factorial-type expressions such as $1.3.5.\ldots.(2n-1)$ or $2.4.6\ldots.2n$]

Procedure: First apply Ratio Test.
In case the Ratio Test fails ($l = 1$), try Comparison Test

ex 1) Test for conv : $\frac{1}{2} + \frac{\lfloor 2}{8} + \frac{\lfloor 3}{32} + \cdots$

Soln the series is : $\frac{\lfloor 1}{2^1} + \frac{\lfloor 2}{2^3} + \frac{\lfloor 3}{2^5} + \cdots$

$$u_n = \frac{\lfloor n}{2^{2n-1}} \qquad u_{n+1} = \frac{\lfloor n+1}{2^{2n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{\lfloor n}{2^{2n-1}} \cdot \frac{2^{2n+1}}{\lfloor n+1} = \frac{4}{n+1}$$

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 0 < 1$ $\quad\therefore \sum u_n$ diverges by Ratio Test.

ex 2) Test for conv : $\sum_{n=1}^{\infty} \frac{\lfloor n}{n^n}$

Soln. $u_n = \frac{\lfloor n}{n^n} \qquad u_{n+1} = \frac{\lfloor n+1}{(n+1)^{n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{\lfloor n}{n^n} \cdot \frac{(n+1)^{n+1}}{\lfloor n+1} = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{1}{n+1} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n$$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = e > 1$ $\quad\therefore \sum u_n$ converges by Ratio Test.

ex 3) Test for conv. $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \cdots$

Soln. $u_n = \frac{\lfloor n}{3.5.7\ldots(2n+1)} \qquad u_{n+1} = \frac{\lfloor n+1}{3.5.7\ldots(2n+3)}$

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{\lfloor n}}{3.5.7\cdots(2n+1)} \cdot \frac{3.5.7\ldots\ldots(2n+3)}{\lfloor n+1 \; n+1} = \frac{2n+3}{n+1}$$

$$= \frac{\cancel{n}\left(2 + \frac{3}{n}\right)}{\cancel{n}\left(1 + \frac{1}{n}\right)}$$

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{2}{1} = 2 > 1$ $\quad\therefore \sum u_n$ converges by Ratio Test

ex 4) Test for conv : $x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots\cdots$ $\quad (x > 0)$

Soln: $u_n = \frac{x^{2n-1}}{\lfloor 2n-1} \qquad u_{n+1} = \frac{x^{2n+1}}{\lfloor 2n+1}$

$$\frac{u_n}{u_{n+1}} = \frac{x^{2n-1}}{\lfloor 2n-1} \cdot \frac{\lfloor 2n+1}{x^{2n+1}} = \frac{1}{x^2} \cdot 2n(2n+1)$$

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty > 1$ $\quad\therefore \sum u_n$ converges by Ratio Test

Ex 4) Test for conv. $\frac{x}{\sqrt{5}} + \frac{x^3}{\sqrt{7}} + \frac{x^5}{\sqrt{9}} + \dots\dots\dots$ $(x>0)$ (11)

Soln. $u_n = \frac{x^{2n-1}}{\sqrt{2n+3}}$ $\quad u_{n+1} = \frac{x^{2n+1}}{\sqrt{2n+5}}$

$$\frac{u_n}{u_{n+1}} = \frac{x^{2n-1}}{\sqrt{2n+3}} \frac{\sqrt{2n+5}}{x^{2n+1}} = \frac{1}{x^2} \frac{\sqrt{n}\sqrt{2+\frac{5}{n}}}{\sqrt{n}\sqrt{2+\frac{3}{n}}}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

Case (i) $x < 1$

$\therefore x^2 < 1$ $\quad (\because x>0)$

$\therefore \frac{1}{x^2} > 1$

$\therefore$ by Ratio Test, Σu_n converges

Case (ii) $x > 1$

$\therefore x^2 > 1$

$\therefore \frac{1}{x^2} < 1$

by Ratio Test, Σu_n diverges

Case (iii): $x = 1$

In this case, $u_n = \frac{1}{\sqrt{2n+3}} = \frac{1}{\sqrt{n}\sqrt{2+\frac{3}{n}}}$

Take $v_n = \frac{1}{\sqrt{n}}$

$\frac{u_n}{v_n} = \frac{1}{\sqrt{2+\frac{3}{n}}}$ $\quad \therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{\sqrt{2}}$ (finite & non-zero)

$\therefore$ by Comparison Test, Σu_n & Σv_n have same behaviour.

Also, Σv_n diverges (as $p = \frac{1}{2} < 1$)

$\therefore \Sigma u_n$ also diverges

Ans: $x < 1 \Rightarrow \Sigma u_n$ converges

$x \geq 1 \Rightarrow \Sigma u_n$ diverge

ALTERNATING SERIES

DEFN An alternating series is of the form

$$u_1 - u_2 + u_3 - u_4 + \cdots \qquad (u_n > 0 \ \forall n \in N)$$

ie $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$

LEIBNITZ' TEST

Let $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ be an alternating series (ie $u_n > 0 \ \forall n \in N$)

Suppose that (i) $\langle u_n \rangle$ is $\downarrow$ (ie $u_1 > u_2 > u_3 > \cdots$)

ii) $\lim_{n\to\infty} u_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges

PROBLEM SET 5 (Alternating Series)

ex(1)

$$\frac{1}{1} - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \cdots$$

given series is $\sum (-1)^{n-1} u_n$; where $u_n = \frac{1}{4n-3}$

(i) $u_{n+1} = \frac{1}{4(n+1)-3} = \frac{1}{4n+1}$

$4n-3 < 4n+1 \Rightarrow \frac{1}{4n-3} > \frac{1}{4n+1}$ ie $u_n > u_{n+1}$

$\therefore \langle u_n \rangle$ is $\downarrow$

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{4n-3} = 0$

by Leibnitz Test $\sum (-1)^{n-1} u_n$ converges

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ex(2)

$$\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \cdots$$

given series is $\sum (-1)^{n-1} u_n$, where $u_n = \frac{1}{(2n-1)2n}$

(i) $u_{n+1} = \frac{1}{(2n+1)(2n+2)}$

$\left.\begin{array}{l} 2n-1 < 2n+1 \\ 2n < 2n+2 \end{array}\right\} \Rightarrow (2n-1)2n < (2n+1)(2n+2)$

$\Rightarrow \frac{1}{(2n-1)2n} > \frac{1}{(2n+1)(2n+2)}$

ie $u_n > u_{n+1}$

$\therefore \langle u_n \rangle$ is $\downarrow$

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n-1)2n} = 0$

$\therefore$ by Leibnitz Test, $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ Converges ex 3) given series is $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$, where $u_n = \dfrac{1}{\sqrt{n^5} + \sqrt{(n+1)^5}}$

(i) $u_{n+1} = \dfrac{1}{\sqrt{(n+1)^5} + \sqrt{(n+2)^5}}$

$$\left.\begin{aligned} \sqrt{n^5} &< \sqrt{(n+1)^5} \\ \sqrt{(n+1)^5} &< \sqrt{(n+2)^5} \end{aligned}\right\} \Rightarrow \sqrt{n^5} + \sqrt{(n+1)^5} < \sqrt{(n+1)^5} + \sqrt{(n+2)^5}$$

$$\Rightarrow \frac{1}{\sqrt{n^5} + \sqrt{(n+1)^5}} > \frac{1}{\sqrt{(n+1)^5} + \sqrt{(n+2)^5}}$$

ie $u_n > u_{n+1}$.

$\therefore$ $\langle u_n \rangle$ is $\downarrow$

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \dfrac{1}{\sqrt{n^5} + \sqrt{(n+1)^5}} = 0$

$\therefore$ by Leibnitz Test, $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges.

ex 4) given series is $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ where $u_n = \dfrac{n+1}{2n}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \dfrac{\cancel{n}(1+\frac{1}{n})}{2\cancel{n}} = \dfrac{1}{2} \neq 0$

$\therefore$ clearly, $\lim_{n\to\infty} (-1)^{n-1} u_n \neq 0$

$\therefore$ series does not converge.

ex 5) Show that the series

$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \cdots \quad \text{converges}$$

Soln Given series is $\sum_{n=2}^{\infty} (-1)^n u_n$, where $u_n = \frac{\log n}{n^2}$

(i) We show $\langle u_n \rangle$ is $\downarrow$

Consider $u(x) = \frac{\log x}{x^2}$ on $[2, \infty[$

$$u'(x) = \frac{x^2 \cdot \frac{1}{x} - \log x \cdot 2x}{x^4} = \frac{1 - 2\log x}{x^3}$$

Now $u'(x) < 0 \iff 1 - 2\log x < 0$

$\iff \log x > \frac{1}{2}$

$\iff x > e^{1/2}$, ie $x > \sqrt{e}$

$\therefore u(x) \downarrow$ on $[2, \infty[$

In particular, $u(n+1) \leq u(n)$ $\forall\, n = 2, 3, 4, \ldots$

ie, $u_{n+1} \leq u_n$

(ii) We now show $\lim_{n \to \infty} u_n = 0$:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\log n}{n^2} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{1/n}{2n} \quad \text{[by L'Hopital's Rule]}$$

$$= \lim_{n\to\infty} \frac{1}{2n^2}$$

$$= 0$$

$\therefore$ by Leibnitz Test, $\sum_{n=2}^{\infty} (-1)^n u_n$ converges

ARBITRARY SERIES ($u_n \in \mathbb{R}$)

Cauchy's Gen. Principle of Convergence

$\sum u_n$ converges iff for each $\epsilon > 0$, $\exists\, n_0 \in N$ st

$$|u_{n_0+1} + u_{n_0+2} + \cdots + u_n| < \epsilon \quad \forall\, n \geq n_0$$

Proof $\sum u_n$ converges

$\Leftrightarrow$ its S.O.P.S $\langle S_n \rangle$ converges

$\Leftrightarrow$ $\langle S_n \rangle$ is Cauchy [by Cauchy's Conv. Criterion for sequence

$\Leftrightarrow$ for each $\epsilon > 0$, $\exists\, n_0 \in N$ st $|S_n - S_{n_0}| < \epsilon \quad \forall\, n \geq n_0$

$\Leftrightarrow$ " " " " $|u_{n_0+1} + u_{n_0+2} + \cdots + u_n| < \epsilon \quad \forall\, n \geq n_0$

DEFN: $\sum u_n$ is said to be absolutely convergent iff $\sum |u_n|$ converges (& not abs. conv iff $\sum |u_n|$ diverges

Theorem. Absolute Convergence $\Rightarrow$ Convergence

Proof. Given $\sum u_n$ is abs convgt, ie $\sum |u_n|$ converges

$\therefore$ by Cauchy's Gen Princ. of Conv. for each $\epsilon > 0$, $\exists\, n_0 \in N$ st :

$$\big|\, |u_{n_0+1}| + |u_{n_0+2}| + \cdots + |u_n| \,\big| < \epsilon \quad \forall\, n \geq n_0$$

ie $|u_{n_0+1}| + |u_{n_0+2}| + \cdots + |u_n| < \epsilon$ " —— (1)

Consider $|u_{n_0+1} + u_{n_0+2} + \cdots + u_n|$

$\leq |u_{n_0+1}| + |u_{n_0+2}| + \cdots + |u_n|$

$< \epsilon \quad \forall\, n \geq n_0$ by (1)

$\therefore$ by Cauchy's Gen Princ. of Conv, $\sum u_n$ converges

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Note: Converse is not true

example of convergent but not abs. convgt.

Consider $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Convergent The above series is $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} u_n$, where $u_n = \frac{1}{n}$

(i) Clearly $u_{n+1} < u_n$ $\quad \langle u_n \rangle \downarrow$

(ii) $\displaystyle\lim_{n\to\infty} u_n = 0$

$\therefore$ by <u>Leibnitz Test</u>, $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges

<u>Not abs. convgt</u> :

Consider $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$

This diverges as $p = 1$.

$\therefore \sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is <u>not</u> abs. convgt.

Defn. A series $\sum u_n$ is s.t.b <u>Conditionally convgt</u> iff it is convergent but not absolutely convgt.

ex) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

(ex1) Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ is absolutely convgt if $p > 1$ & conditionally convgt if $p \leq 1$

<u>Solution</u> <u>Test for absolute conv</u> :

Consider $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{n^p} \right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$

which converges if $p > 1$
& diverges if $p \leq 1$.

$\therefore \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ is <u>abs Convgt</u> if <u>$p > 1$</u>
& <u>not abs convgt</u> if <u>$p \leq 1$</u>

<u>Test for convergence</u> :

Consider $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p} = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$

(i) Clearly $\langle u_n \rangle \downarrow$

(ii) Clearly $\lim_{n \to \infty} u_n = 0$

$\therefore$ by <u>Leibnitz Test</u>, $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges.

Thus $p > 1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ is abs. convgt

& $p \leq 1 \Rightarrow$ " " conditionally convgt.

ex 2) Test for convergence & absolute convergence :

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}+1}$ (b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{\lfloor n+1}$

Soln (a) Abs conv

Consider $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{\sqrt{n}+1} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

$u_n = \frac{1}{\sqrt{n}+1} = \frac{1}{\sqrt{n}\left(1+\frac{1}{\sqrt{n}}\right)}$

Take $v_n = \frac{1}{\sqrt{n}}$, we get (by Comp. Test) that $\sum \frac{1}{\sqrt{n}+1}$ divrg

ie $\sum (-1)^{n-1} \frac{1}{\sqrt{n}+1}$ is not abs convgt

Convgt.

Using Leibnitz Test, we get $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}+1}$ converges (Verify !)

Thus, given series is convgt. but not abs convgt (ie conditionally convgt)

(b) Abs conv. :

Consider $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{n^2}{\lfloor n+1} \right| = \sum_{n=1}^{\infty} \frac{n^2}{\lfloor n+1}$,

$u_n = \frac{n^2}{\lfloor n+1}$

Using D'Alembert's Ratio Test, we get $\sum u_n$ converges (Verify !)

$\therefore \sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is abs. convgt

Convgt :

As it is abs convgt $\therefore$ it is also convgt

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ex 3) Test for convergence & absolute convergence

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{\cos n\alpha}{n\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n\alpha}{n^3}$

Soln (a) Abs conv.

Consider $\sum_{n=1}^{\infty} \left| (-1)^n \frac{\cos n\alpha}{n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{|\cos n\alpha|}{n^{3/2}}$

Let $u_n = \frac{|\cos n\alpha|}{n^{3/2}} \leq \frac{1}{n^{3/2}}$

Also $\sum \frac{1}{n^{3/2}}$ converges $[\because p = 3/2 > 1]$

$\therefore$ by Basic Comp Test, $\sum u_n$ also conv.

MISC. PROBLEMS IN INFINITE-SERIES (USING BCT)

ex 1) Test for convergence : (1) $\sum_{n=3}^{\infty} \frac{1}{n^2 \log n}$ (2) $\sum_{n=2} \frac{1}{\log n}$

Soln (1) Now $\log n \geq 1$ $\forall n \geq 3$ [Note !]

$\frac{1}{\log n} \leq 1$ "

$\frac{1}{n^2 \log n} \leq \frac{1}{n^2}$ "

Now $\sum \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore$ by Basic Comp Test $\sum u_n$ also converges

(2) We know that :

$\log n < n$ $\forall n \in \mathbb{N}$

$\therefore \frac{1}{\log n} > \frac{1}{n}$ "

$\sum \frac{1}{n}$ diverges [$\because p = 1$]

$\therefore \sum \frac{1}{\log n}$ also diverges [by B.C.T]

ex 2) Test for convergence $\sum_{n=1}^{\infty} \frac{\sin nx + \cos nx}{n^{3/2}}$

Soln Let $u_n = \frac{\sin nx + \cos nx}{n^{3/2}}$

$\therefore |u_n| = \frac{|\sin nx + \cos nx|}{n^{3/2}} \leq \frac{|\sin nx| + |\cos nx|}{n^{3/2}}$

Now $\sum \frac{1}{n^{3/2}}$ converges (as $p = \frac{3}{2} > 1$)

$\therefore$ by Basic Comp. Test, $\sum |u_n|$ also converges

ie $\sum u_n$ converges absolutely

$\therefore \sum u_n$ converges [$\because$ abs. conv $\Rightarrow$ conv.]

For any additional information or discussion, students may feel free to contact the undersigned.

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